Strategic voting and the logic of knowledge

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Abstract

We propose a general framework for strategic voting when a voter may have lack knowledge about other votes or about other voters' knowledge about her own vote. In this setting we define notions of manipulation and equilibrium. We give a case study for two voters and plurality voting, and for three voters and Borda voting. We also model knowledge changing actions, such as revealing your voting preference and declaring your vote.

1 Introduction

A well-known fact in social choice theory is that strategic voting, also known as manipulation, becomes harder when voters know less about the preferences or votes of other voters. Standard approaches to manipulation in social choice theory [6] as well as in computational social choice [3] assume that the manipulating voter or the manipulating coalition knows perfectly how the other voters will vote. Some approaches [2] assume that voters have a probabilistic prior belief on the outcome of the vote, which encompasses the case where each voter has a probability distribution over the set of profiles. A recent paper [5] extends coalitional manipulation to incomplete knowledge, by distinguishing manipulating from non-manipulating voters and by considering that the manipulating coalition has, for each voter outside the coalition, a set of possible votes encoded in the form of a partial order over candidates. Uncertainty of voters about the uncertainties of other voters, i.e., higher-order beliefs of voters, has not been treated in full generality.

We model how uncertainty about the preferences of other voters may determine a strategic vote, and how a reduction in this uncertainty may change a strategic vote. A link between epistemic logic and voting has been given in [4]—they use knowledge graphs

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to indicate that a voter is uncertain about the preference of another voter. A more recent approach, within the area known as social software, is [8]. The recent [5] walks a middle way namely where equivalence classes are called information sets, as in treatments of knowledge and uncertainty in economics, but where the uncertain voter, or coalition, does not take the uncertainty of other voters into account.

2 Knowledge and Voting

We assume *voters* $\mathcal{N} = \{1, \ldots, n\}$, *candidates* $\mathcal{C} = \{a, b, c, \ldots\}$, and *votes* $V_i \subseteq \mathcal{C} \times \mathcal{C}$ that are linear orders. If agent i prefers candidate a to candidate b, we write $a \succ_i b$. A *profile* P is a collection $\{V_1, \ldots, V_n\}$ of n votes, and a *voting rule* is a function $F : O(\mathcal{C})^n \to \mathcal{C}$ from the set of profiles to the set of candidates. We may further assume a tie-breaking mechanism. If $F(P[V_i/V_i']) \succ_i F(P)$, then V_i' is a *successful manipulation*. Given a profile P, a profile P' is an *equilibrium profile* iff no agent has a successful manipulation.

We model uncertainty about voting as incomplete knowledge about profiles. This terminology is standard in modal logic. The novelty consists in taking models with *profiles* instead of *valuations of propositional variables.*

Definition 1 (Knowledge profile) *A* profile model *is a structure* $\mathcal{P} = (S, \{\sim_1, \ldots, \sim_n\}, \pi)$*, where* S *is a* domain *of abstract objects called* profile names*; where for* $i = 1, \ldots, n$*,* $\sim_i$ *is an* indistinguishability relation*, that is, an equivalence relation; and where* valuation $\pi : S \to O(\mathcal{C})^n$ *assigns a profile to each profile name. A* knowledge profile *is pointed structure* $\mathcal{P}_s$ *where* $\mathcal{P}$ *is a profile model and* s *is a profile name in the domain of* $\mathcal{P}$*.*

Definition 2 (Knowledge) *Given a knowledge profile* $\mathcal{P}_s$ *and a proposition* q*,* agent i knows that q *if and only if* q *holds for all profile names in* $\mathcal{P}$ *indistinguishable for* i *from* s *(i.e., for all* $s' \in \mathcal{P}$ *such that* $s \sim_i s'$*).*

Propositions like 'voter i knows the profile' or even 'voter i knows that P is an equilibrium profile' have a precise formal description in this framework.

Under conditions of incomplete knowledge it may be that voter i (or coalition G) can manipulate the outcome of a profile P but does not know that, because she considers another profile (name) possible that she cannot manipulate. Such situations call for more refined notions of manipulation, that also involve knowledge. They can be borrowed from the knowledge and action literature [9, 7]. Our main interest is when voters know the manipulation.

Definition 3 (Knowledge of manipulation) *Given a knowledge profile* $\mathcal{P}_s$*. Voter* i *knows* **de re** *that she can* strongly successfully *manipulate* $\mathcal{P}_s$ *if there is a vote* V_i' *such that for all* t *such that* $s \sim_i t$*,* $F(P[V_i/V_i']) \succ_i F(P)$*, where* t *has profile* P*.*

In the presence of knowledge, the definition of an equilibrium extends naturally. The trick is that for each agent, the combination of an agent i and an equivalence class $[s]_{\sim_i}$ for that agent (for some state s in the knowledge profile) defines a *virtual agent*. Thus, agent i

is multiplied in as many virtual agents as there are equivalences classes for $\sim_i$ in the model. An equilibrium is then a combination of votes such that none of the virtual agents has an interest to deviate. An intuitively more appealing solution than virtual agents, applied in [1], is to stick to the agents we already have, but change the set of votes into a larger set of *conditional votes* — where the conditions are the equivalence classes for the agents. This we will now follow in the definition below. For *risk averse* voters (this criterion fits best our probability-free and utility-free model — it was also chosen in [5]) we can effectively determine if a conditional profile is an equilibrium without taking probability distributions into account, unlike in the more general setting of Bayesian games that it originates with.

Definition 4 (Conditional equilibrium) *Given is a knowledge profile model $\mathcal{P}$. For each agent i, let CV_i be the set of all* conditional votes *for that agent. A* conditional vote *is a function $CV_i : S/{\sim_i} \to O(\mathcal{C})$, i.e., a function that assigns to each equivalence class for that agent a vote. A* conditional profile *is a collection of n conditional votes, one for each agent. A conditional profile is an* equilibrium *iff no agent has a successful manipulation. A conditional profile is a* strong equilibrium *iff no coalition has a successful manipulation.*

3 Example: plurality voting

Consider two voters $1, 2$, four candidates a, b, c, d, and three profile names s, t, u (for two profiles P and P') as below. The profile name s is assigned to profile P, wherein $a \succ_1 c \succ_1 b \succ_1 d$ and $d \succ_2 c \succ_2 b \succ_2 a$, etc. Profile names that are indistinguishable for a voter i are linked with an i-labelled edge. The partition for 1 on the domain is therefore $\{\{s, t\}, \{u\}\}$, and the partition for 2 on the domain is $\{\{s\}, \{t, u\}\}$.

1	2		1	2		1	2
a	d		a	d		d	d
c	c	—1—	c	c	—2—	c	c
b	b		b	b		b	b
d	a		d	a		a	a
s, P			t, P			u, P'	

Note that the names s and t are assigned to the same profile. However, s and t have different epistemic properties. In s, 2 knows that 1 prefers a over d, whereas in t 2 does not know that.

Consider a plurality vote with a tie-breaking rule $b \succ a \succ c \succ d$. If there had been no uncertainty, then in profile P, if 1 votes for her preference a and 2 votes for his preference d, then the tie prefers a, 2's least preferred candidate. If instead 2 votes c, a will still win. But if 2 votes b, b wins. We observe that (a, b) and (b, b) are equilibria pairs of votes, and that for 1 voting a is dominant. If there had been no uncertainty, then in profile P' pair (d, d) is the dominant equilibrium.

This situation changes when we take the uncertainty of the voters into account. There are two equilibria that we can associate with this knowledge profile model. Below, the conditional vote for 1 in the first equilibrium actually is defined as (given that $\pi(t) = P$ and $\pi(u) = P'$): $CV_1(\{t\}) = V_1$ and $CV_1(\{u\}) = V_1'$; the vote for 2 is conditional to one equivalence class — in other words, it is unconditional. The equivalent verbose formulation is more intelligable:

- (if 1 prefers a then 1 votes a and if 1 prefers d then 1 votes d, 2 votes b),
- (if 1 prefers a then 1 votes b and if 1 prefers d then 1 votes d, 2 votes b).

Unfortunately for voter 2, if the actual profile is P' so that d is his equilibrium vote, he will still not be inclined to cast that vote because he considers it possible that the profile is P, where, if 2 votes d and 1 votes a, a gets elected, voter 2's least preferred candidate. As 2 is risk averse his (known) equilibrium vote is therefore b.

If P' is the case, voter 1 has an incentive to make her true vote (i.e., her intention) known to 2, and even to declare her vote prior to 2.

4 Example: de dicto knowlege with Borda voting

We now consider manipulation with voting according to the Borda voting rule. Consider three agents, four candidates, and two profiles P and P' that are indistinguishable for agent 1, but that agents 2 and 3 can tell apart; as follows.

1	2	3		1	2	3
c	d	b		c	d	b
b	a	d	1	b	a	a
a	c	c		a	c	c
d	b	a		d	b	d

P $\qquad$ P'

There is also a tie-breaking preference $b \succ c \succ d \succ a$. The only difference between the profiles P and P' is that 3 prefers d over a in P but a over d in P'. We prove that 1 can manipulate the election if the profile is P, and that 1 can manipulate the election if the profile is P'. But the manipulations are different for the different profiles. Even worse: the manipulation for P gives a worse outcome for P', and the manipulation for P' gives a worse outcome for P. Therefore she is not effectively able to manipulate the outcome of the election. (In the literature, this tends to be called *de dicto* knowledge [9, 7], in our case therefore: de dicto knowledge of manipulation, in contrast with the *de re* knowledge of manipulation defined above.)

In Borda, the ranks for each candidate in each vote are added up, and the candidate with the highest sum wins, modulo the tie-breaking preference. The preferred candidate gets 3 points, the 2nd choice 2 points, etc. First, the outcome when all three agents give their true vote. We write $xyzw$ when there are x points for a, y for b, z for c, w for d.

profile	count	observation	outcome
P	3555	b, c, d are tied	b
P'	5553	a, b, c are tied	b

1 can manipulate P or P' by downgrading b. But this is tricky, because it comes at the price of making a or d, or both, more preferred. We now see that this price is too high.

In P, 1 can achieve a better outcome by V_1' defined as $1 : cabd$. Let $Q = P[V_1/V_1']$, and $Q' = P[V_1/V_1']$. Although 1 prefers the winner in Q over the winner in P, the winner in Q' is less preferred by her than the winner in P':

profile	count	observation	outcome
Q	4455	c, d are tied	c
Q'	6453		a

In P', 1 can achieve a better outcome by V_1'' defined as $1 : cdba$. Let $R = P[V_1/V_1'']$, and $R' = P[V_1/V_1'']$. Now, dually to the above, 1 prefers the winner in R' over the winner in P', but the winner in R is less preferred by her than the winner in P:

profile	count	observation	outcome
R	2457	1's worst dream	d
R'	4455	c, d are tied	c

For the convenience of the reader wishing to check all such results we present them for all different votes for voter 1 where c is most preferred. (As said above, making c less preferred is not successful.) The first row gives the outcome of the profile wherein 1's true vote in P is replaced by the successive votes in the columns (the first column is the true vote); the second row, the same but for P'.

$1 : cbad$	$1 : cabd$	$1 : cdba$	$1 : cadb$	$1 : cdab$	$1 : cbda$
$b(3555)$	$c(4455)$	$d(2457)$	$d(4356)$	$d(3357)$	$d(2556)$
$b(5553)$	$a(6453)$	$c(4455)$	$a(6354)$	$c(5355)$	$b(4554)$

5 Dynamics

The modal logical setting for voting and knowledge can be extended with dynamic logical operations. Three examples are: *deliberation of a coalition*, *public announcement of a proposition* (such as an agent revealing her true preference), and *declaring a vote*. These can be formalized as semantic operations $P_s \mapsto P_s|G$, $P_s \mapsto P_s|p$ (for proposition p), and $P_s \mapsto P_s|\mathsf{d}(V_i)$, respectively. All these correspond to standard dynamic epistemic logical operations [10].

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