FM–representability and Computable Games as Learning Algorithms

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This paper is based on research in the area of mathematics without actual infinity as started by Jan Mycielski in [Myc81] and further developed by Mostowski in [Mos01], [Mos03] and numerous others. The general aim of this research is giving an answer to the question what mathematical tools are available, if only finite but potentially infinite domains are considered. Potential infinity here stands in opposition to actual infinity. Both of these notions were introduced by Aristotle and later popularized in the area of mathematics by David Hilbert in [Hil26]. An actually infinite set is a collection consisting of infinitely many elements that can be accessed at once. A potentially infinite set is a collection of only finitely many elements that can always be expanded by additional ones, according to some well defined rule.

In [Mos01] Mostowski obtained a result stating which mathematical relations can be meaningfully described in potentially infinite world. It turned out that these relations are exactly the relations that are algorithmically learnable, as defined independently by Putnam in [Put65] and Gold in [Gol65] and [Gol67]. This result was in favor of philosophical intuitions formulated by Aristotle, according to which the notion of potential infinity is sufficient to describe the whole humans' cognition. From the technical point of view, the result made it possible to use tools developed in the course of studying algorithmic learnability to do research on potentially intinite mathematics and vice versa.

One natural way to further develop Mostowski's result was to find other ways to describe the set of relations representable in potentially infinite domain. In this paper we formulate a type of games that serve this purpose.

Mostowski used the notions of finite models domain, shortly FM–domain, and mathematical representability in finite models, shortly FM–representability, to prove his result. These notions are defined as follows.

Definition 1. *Let $\mathbb{N}$ be the standard arithmetical model, i.e. $(\{0,1,2,\ldots\},+,\times,0)$. Let $\mathbb{N}_n$ be a submodel of $\mathbb{N}$ consisting of n elements, i.e. $(\{0,\ldots,n-1\},+^{n-1},\times^{n-1},0,n-1)$, where $+^{n-1}$ and $\times^{n-1}$ are the well known arithemtical operations treated as ternary relations (since they will obviously be undefined for some inputs in a finite domain) whose arguments are at most $n-1$. We additionally distinguish the $n-1$–th element to be able to easiliy refer to the maximal element.*

We say that the finite models domain of $\mathbb{N}$, or $FM(\mathbb{N})$, is the family of models $\mathbb{N}_n$, for all n.

Definition 2. *Let $R \subseteq \mathbb{N}^n$. We say that the formula ϕ FM–represents R if for every $a_1,\ldots,a_n$ the following conditions are true:*

- *if $R(a_1,\ldots,a_n)$, then there is a k such that for all $t>k$ $\mathbb{N}_t \models \phi(a_1,\ldots,a_n)$*
- *if $\neg R(a_1,\ldots,a_n)$, then there is a k such that for all $t>k$ $\mathbb{N}_t \models \neg\phi(a_1,\ldots,a_n)$,*

where $\mathbb{N}_n$ is the finite model $(\{0,\ldots,n-1\},+^{n-1},\times^{n-1})$ from $FM(\mathbb{N})$.

We then say that a relation R is FM–representable if it is FM–represented by some formula ϕ.

The result from Mostowski is the following.

Theorem 1. *Let R be a realation on natural numbers. Then the following are equivalent:*

- *R is Δ^0_2 in the arithmetical hierarchy*
- *R is recursive with recursively enumerable oracle*
- *R is of degree $\leq 0'$ in terms of Turing degrees*

– *R is algorithmically learnable*
– *R is FM–representable*

□

The notion of algorithmic learnability used in the above theorem comes from Putnam and Gold who defined it as follows.

Definition 3. *Let $R \subseteq \mathbb{N}^n$. A recursive function $f\colon \mathbb{N}^{n+1} \to \mathbb{N}$ is said to be a learning function for the relation R, if for any $a_1, \ldots, a_n$ the following is true:*

– *there is an s such that for all $t > s$ $f(a_1, \ldots, a_n, t) = 1$ iff $R(a_1, \ldots, a_n)$ is true*
– *there is an s such that for all $t > s$ $f(a_1, \ldots, a_n, t) = 0$ iff $R(a_1, \ldots, a_n)$ is false*

We say that R is learnable iff there is a recursive function f that learns R.

Since all the learning functions are computable, one can think of them as Turing machines that work on inputs $a_1, \ldots, a_n$ and on additional input t, which is meant to be the indicator of time spent on learning. For any t starting with 0 such a machine is started and it gives the output "yes" iff it thinks that $a_1, \ldots, a_n \in R$ and it gives the output "no" otherwise. Then the input t is incremented and the machine is started again. If after some finite number of runs the answers of the machine stabilize (i.e. they don't change ever after), then this is a learning machine for R. If they don't stabilize, then this is not a learning machine for R.

As an example, we can take a machine that takes two inputs: a first order formula ϕ (or its Gödel number, to be precise) and a natural number t, and returns "yes" if there is a first–order proof of ϕ of length t and "no" otherwise. The answers of this machine will obviously stabilize — they will always be "no", if a particular ϕ is not a first–order theorem and otherwise there will be some k such that ϕ has a proof of length $k + n$, for any n (since it's trivially true that if ϕ has a proof of lenth k, then it has a proof of lenth $k + 1$). Therefore, the relation of being a provable first–order logic formula is learnable.

In order to describe the set of FM–representable relations and at the same time learnable relations in terms of Game Theory, we propose the following type of game. There are two players: PLAYER 1 and PLAYER 2 who are given a first–order arithmetical formula $\varphi(x_1, \ldots, x_k)$ with k free variables and a k–tuple of natural numbers $a_1, \ldots, a_k$.

PLAYER 1 chooses a natural number n_1 such that the sentence $\varphi(a_1, \ldots, a_k)$ is true in model $\mathbb{N}_{n_1}$ from $FM(\mathbb{N})$.

PLAYER 2 chooses a natural number m_1 greater than n_1 and such that the sentence $\neg\varphi(a_1, \ldots, a_k)$ is true in model $\mathbb{N}_{m_1}$ from $FM(\mathbb{N})$.

PLAYER 1 chooses a natural number n_2 greater than m_1 and such that the sentence $\varphi(a_1, \ldots, a_k)$ is true in model $\mathbb{N}_{n_2}$ from $FM(\mathbb{N})$.

... and so on...

The player who has no possible move loses.

The following holds.

Theorem 2. *Let R be a k–ary realation on natural numbers. Then the following are equivalent:*

– *There is a first–order arithmetical formula $\phi(x_1, \ldots, x_k)$ such that for all k–tuples of natural numbers the game given by the formula and the k–tuple is determined and PLAYER 1 has a winning strategy exactly when the k–tuple is in R.*
– *R is Δ^0_2 in the arithmetical hierarchy*
– *R is recursive with recursively enumerable oracle*
– *R is of degree $\leq 0'$ in terms of Turing degrees*
– *R is algorithmically learnable*
– *R is FM–representable*

Proof. Suppose there is a first–order arithmetical formula $\phi(x_1, \ldots, x_k)$ such that for all k–tuples of natural numbers a game of our type given by the formula ϕ and a fixed k–tuple is determined and PLAYER 1 has a winning strategy exactly when the k–tuple is in R.

Let $a_1, \ldots, a_k$ be natural numbers. If $R(a_1, \ldots, a_k)$ holds, then PLAYER 1 has a winning strategy in m moves, for some m. Therefore, by definition of our game, for all $t > m$ $\mathbb{N}_t \models \phi(a_1, \ldots, a_k)$. If, on the other hand, $R(a_1, \ldots, a_k)$ doesn't hold, then PLAYER 2 has a winning strategy in m moves, for some m and for all $t > m$ $\mathbb{N}_t \models \neg\phi(a_1, \ldots, a_k)$. Therefore $\phi(x_1, \ldots, x_k)$ FM–represents R.

Let's assume conversely that R is FM–represented by some formula $\phi(x_1, \ldots, x_k)$. Let $a_1, \ldots, a_k$ be natural numbers. If $R(a_1, \ldots, a_k)$ holds, then there is an m such that for all $t > m$ $\mathbb{N}_t \models \phi(a_1, \ldots, a_k)$. Therefore in a game of our type given by ϕ and $a_1, \ldots, a_k$ PLAYER 1 has a winning strategy consinsting of one move — he just has to give m and PLAYER 2 has no move. If $R(a_1, \ldots, a_k)$ doesn't hold, then there is an m such that for all $t > m$ $\mathbb{N}_t \models \neg\phi(a_1, \ldots, a_k)$. In a game of our type given by ϕ and $a_1, \ldots, a_k$ PLAYER 2 wins by giving m regardless of what PLAYER 1 does in his first move.$\square$

Mostowski pointed out in [Mos12] that if we add a restriction that the strategies of both players must be recursive, then these games become nontrivial — the number of moves in a determined game will be uncomputable in general case. This is because such recursive strategies can be easily transformed into recursive learning functions and giving a point where such a function stabilizes is a Δ^0_2 problem in general case.

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