Conservativity

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Recall the definition of conservativity.

Definition 1 *A learning function M is* conservative *if, for each σ and x, content$(\sigma^\wedge\langle x\rangle) \subseteq L_{M(\sigma)}$ implies $M(\sigma^\wedge\langle x\rangle) = M(\sigma)$.*

Theorem 1 *There exists a collection of languages that is effectively identifiable, but not by a effective conservative learner.*

Proof Let us first focus on constructing the witness class. Initially, take $\mathcal{L} = \{L_i = \{\langle i, x\rangle \mid x \in \mathbb{N}\} \mid i \in \mathbb{N}\}$. Consider a particular $j \in \mathbb{N}$, the corresponding $L_j \in \mathcal{L}$, and the corresponding φ_j, a partial recursive function. We can think of φ_j as of a (not necessarily successfull) learner. Let us consider a particular text for L_j, $t^j = \langle j, 0\rangle, \langle j, 1\rangle, \langle j, 2\rangle, \langle j, 3\rangle, \ldots$ If φ_j happens to be a function that identifies L_j, then on some initial segment $t^j[n+1]$, φ_j will output an index of L_j, say i (obviously, if φ_j does not identify L_j, this does not have to happen). Moreover, note that enumerating the set W_i will at some point give a superset of $\{\langle j, 0\rangle, \ldots, \langle j, n\rangle\}$ (here we are using the fact that the successful learner must be a total function).

Using this interpretation we will now extend the class $\mathcal{L}$ in the following way. For each $j \in \mathbb{N}$ we add a language L'_j defined in the following way:

$$L'_j = \begin{cases} \{\langle j, 0\rangle, \ldots, \langle j, n\rangle\} & \text{where } \langle n, s\rangle \text{ are the smallest s.t.} \\ & \{\langle j, 0\rangle, \ldots, \langle j, n\rangle\} \subset W_{\varphi_j^s(t^j[n+1]),s}; \\ \{\langle j, 0\rangle\} & \text{if such a pair does not exist.} \end{cases}$$

Assume that an effective, conservative learner M identifies $\mathcal{L}$. We will derive a contradiction.

Such a learner is a recursive function, so M is in fact φ_j for some $j \in \mathbb{N}$. Since φ_j identifies $\mathcal{L}$, it identifies $L_j \in \mathcal{L}$. Then, on text $t^j = \langle j, 0\rangle, \langle j, 1\rangle, \langle j, 2\rangle, \langle j, 3\rangle, \ldots$ there will be a (smallest) pair $\langle n, s\rangle$, that will guarantee the existence of $L'_j = \{\langle j, 0\rangle, \ldots, \langle j, n\rangle\}$ in $\mathcal{L}$. Now consider the text $\langle j, 0\rangle, \ldots, \langle j, n\rangle, \langle j, n\rangle, \langle j, n\rangle, \ldots$ for L'_j. On the first occurrence of $\langle j, n\rangle$, φ_j will output i for L_j, and since the rest of the text does not contradict L_j, φ_j will never change the output (because it is conservative). Hence, φ_j will not identify L'_j. Contradiction.

It remains to be shown that $\mathcal{L}$ is identifiable by a recursive learner M. Consider two cases, depending on the first element seen by M:

1. $\langle j, m\rangle$, with $m \neq 0$, then M will output an index of L_j on any sequence σ extending $\langle j, m\rangle$, unless it is the case that $\{\langle j, 0\rangle, \ldots, \langle j, n\rangle\} \subset W_{\varphi_j^s(t^j[n+1]),s}$ for some $\langle n, s\rangle \leq lh(\sigma)$. If it is so, it can be determined if all elements of σ are members of L'_j (since both σ and L'_j are finite). If that is the case M outputs L'_j and continues doing so as long as all the elements of the input sequence are elements of L'_j. If that is not the case M switches back to L_j.

2. $\langle j, 0\rangle$, then M conjectures L'_j as long as $\langle j, 0\rangle$ is the only pair seen, otherwise M switches to L_j and continues according to the behavior described before.

$\square$

Theorem 2 (Lange, Zeugmann, and Kapur) *There is a uniformly recursive family of languages $\mathcal{L}$, which is effectively identifiable with respect to some indexing for $\mathcal{L}$, but which is not effectively identifiable by a conservative learner with respect to any indexing.*

Proof The idea here is essentially the same as that of the proof of Theorem 1. The extra complexity is due to the need to produce a uniformly recursive family of languages. Let us construct such a family $\mathcal{L}$.

Let t^i be the text $\langle i, 0\rangle, \langle i, 1\rangle, \ldots$. $\mathcal{L}$ contains all languages $L_{\langle i,s\rangle}$ defined as follows:

$$L_{\langle i,0\rangle} = \{\langle i, x\rangle \mid x \in \mathbb{N}\}$$

and for $s \geq 1$,

$$L_{\langle i,s\rangle} = \begin{cases} \{\langle i,0\rangle, \ldots, \langle i,n\rangle\} & \text{if } n \text{ is the smallest s.t.} \\ & \varphi_i^s(t^i[n+1]) \downarrow \\ & \text{and } \{\langle i,0\rangle, \ldots, \langle i,n\rangle\} \subset W_{\varphi_i^s(t^i[n+1]),s}; \\ \{\langle i, x\rangle \mid x \in \mathbb{N}\} & \text{if no such } n \text{ exists.} \end{cases}$$

First let us show that $L_{\langle i,s\rangle}$'s form a uniformly recursive family of languages. For $s \geq 1$, $\langle i, m\rangle \in L_{\langle i,s\rangle}$ iff for all $n < m$ such that $\varphi_i^s(t^i[n+1]) \downarrow$, $\{\langle i,0\rangle, \ldots, \langle i,n\rangle\} \not\subset W_{\varphi_i^s(t^i[n+1]),s}$ is a decidable condition. So there is a uniformly recursive indexing of $\mathcal{L}$.

$\mathcal{L}$ is identified by a recursive function. To prove this, we show that a telltale set for $L_{\langle i,s\rangle}$ in Ł can be recursively enumerated, given $\langle i, s\rangle$. Let

$$\psi(i, 0) \simeq \begin{cases} \langle n, s\rangle & \text{if } \langle n, s\rangle \text{ is the smallest pair s.t.} \\ & \varphi_i^s(t^i[n+1]) \downarrow \\ & \text{and } \{\langle i,0\rangle, \ldots, \langle i,n\rangle\} \subset W_{\varphi_i^s(t^i[n+1]),s}; \\ \uparrow & \text{otherwise.} \end{cases}$$

$$\psi(i, x+1) \simeq \begin{cases} \langle n, s\rangle & \text{if } \langle n, s\rangle \text{ is the smallest pair s.t.} \\ & 1 \leq s < j_1(\psi(i,x)), \varphi_i^s(t^i[n+1])\downarrow \\ & \text{and } \{\langle i, 0\rangle, \ldots, \langle i, n\rangle\} \subset W_{\varphi_i^s(t^i[n+1]),s}; \\ \uparrow & \text{otherwise.} \end{cases}$$

The set $\{x \mid \psi(i,x)\downarrow\}$ is some initial segment of $\mathbb{N}$, and $j_0(\psi(i,x))$ is strictly increasing in x as long as it is defined (given that we are using the Cantor function j for $\langle n, s\rangle$). Let

$$D_i = \{\langle i, n\rangle \mid n = 0 \vee n \leq j_0(\psi(i,x)) + 1 \text{ for some } x \text{ s.t. } \psi(i,x)\downarrow\}.$$

Then for all s, $L_{\langle i,s\rangle} \subset D_i$ if $L_{\langle i,s\rangle}$ is finite and $D_i \subset L_{\langle i,s\rangle}$ otherwise. This implies that $D_i \cap L_{\langle i,s\rangle}$ is a telltale set for $L_{\langle i,s\rangle}$ in $\mathcal{L}_e$. Since D_i is recursively enumerated, so is $D_i \cap L_{\langle i,s\rangle}$.

That $\mathcal{L}$ cannot be identified by a recursive function conservative on $\mathcal{L}$ (with respect to any index) can be proved in a way similar to the proof of Theorem 1. □

Theorem 3 *Let $\mathcal{L}$ be a uniformly recursive family of languages. There is a total recursive function G that maps each $i \in \mathbb{N}$ to a telltale set $G(i)$ for L_i in $\mathcal{L}$ iff $\mathcal{L}$ is identifiable by a recursive function conservative on $\mathcal{L}$.*

Proof

[⇒]

Define M as follows:

$$M(\sigma) \simeq \begin{cases} \mu i \leq lh(\sigma)(G(i) \subseteq \mathit{content}(\sigma) \subseteq L_i) & \text{if defined;} \\ \uparrow & \text{otherwise.} \end{cases}$$

Then M is conservative on $\mathcal{L}$, and identifies $\mathcal{L}$.

[⇐]

We leave the proof of this direction for later.

□