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TRIAL AND ERROR PREDICATES AND THE SOLUTION TO A PROBLEM OF MOSTOWSKI*

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§ 1. Introduction. The purpose of this paper is to present two groups of results which have turned out to have a surprisingly close interconnection. The first two results (Theorems 1 and 2) were inspired by the following question: we know what sets are "decidable" — namely, the recursive sets (according to Church's Thesis). But what happens if we modify the notion of a decision procedure by (1) allowing the procedure to "change its mind" any finite number of times (in terms of Turing Machines: we visualize the machine as being given an integer (or an n-tuple of integers) as input. The machine then "prints out" a finite sequence of "yesses" and "nos". The *last* "yes" or "no" is always to be the correct answer.); and (2) we give up the requirement that it be possible to tell (effectively) if the computation has terminated? I.e., if the machine has most recently printed "yes", then we know that the integer put in as input must be in the set *unless the machine is going to change its mind*; but we have no procedure for telling whether the machine will change its mind or not.

The sets for which there exist decision procedures in this widened sense are decidable by "empirical" means — for, if we always "posit" that the most recently generated answer is correct, we will make a finite number of mistakes, but we will eventually get the correct answer. (Note, however, that even if we have gotten to the correct answer (the end of the finite sequence) we are never *sure* that we have the correct answer.)

Instead of requiring that the sequence of "yesses" and "nos" be finite and non-empty, we may require that it should always be infinite, but that it should consist entirely of "yesses" (or entirely of "nos") from a certain point on: the class of predicates obtained (which we call the class of "trial and error" predicates), is easily seen to be unchanged.[1] We thus arrive at the following reformulation of our first question: first define

DEFINITION. P is a *trial and error predicate* if and only if there is a g.r. (general recursive) function f such that (for every $x_1, x_2, \ldots, x_n$)

$$P(x_1, x_2, \ldots, x_n) \equiv \lim_{y\to\infty} f(x_1, x_2, \ldots, x_n, y) = 1,$$

$$\bar{P}(x_1, x_2, \ldots, x_n) \equiv \lim_{y\to\infty} f(x_1, x_2, \ldots, x_n, y) = 0,$$

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[1] For going from a finite sequence to an infinite sequence (with repetitions) cf. the last paragraph of the proof of Theorem 3, below. Going in the other direction it

where

$$\lim_{y\to\infty} f(x_1, x_2, \ldots, x_n, y) = k =_{\mathrm{df}} (\mathrm{E}y)(z)(z \geqq y \supset f(x_1, x_2, \ldots, x_n, z) = k).$$

It is obvious that a trial and error predicate is always arithmetical. How non-constructive can such a predicate be? In other words

Question 1: What are necessary and sufficient conditions (in terms of the Kleene-Mostowski Hierarchy of arithmetical predicates) that P be a trial and error predicate?

It is obviously better if we know, not just that P is a trial and error predicate, but that (by using a suitable program) we can keep our machine from ever having to change its mind more than k times (for some fixed k, independent of the particular $x_1, x_2, \ldots, x_n$ about which we are asking — i.e., k is independent of the input). To make this precise, call a predicate P a *k-trial predicate* if there is a g.r. function f and a fixed integer k such that[2] (for all $x_1, x_2, \ldots, x_n$)

(1) $\mathrm{P}(x_1, \ldots, x_n) \equiv \lim_{y\to\infty} f(x_1, x_2, \ldots, x_n, y) = 1,$

(2) There are *at most* k integers y such that
$f(x_1, x_2, \ldots, x_n, y) \neq f(x_1, x_2, \ldots, x_n, y+1).$

Since a k-trial predicate is obviously arithmetical, we are led to ask how non-constructive such a predicate can be, i.e.,

Question 2: What are necessary and sufficient conditions that there exist a k such that P is a k-trial predicate?

Our second group of results is connected with the meta-theory of quantification theory (without identity). A number of years ago, Mostowski[3] reported on his unsuccessful attempts to find a consistent formula of quantification theory with no model in the "smallest field of sets containing the recursively enumerable sets". Since "set" here means set of n-tuples, what Mostowski wanted is, in our terminology, a formula with no model in which (1) the universe of discourse is the natural numbers; and (2) the predicate letters are all interpreted as r.e. (recursively enumerable) predicates, or truth functions (i.e. Boolean combinations) of r.e. predicates.

The main result of § 3 is: a formula of this kind (the kind wanted by Mostowski) does not exist. *Every* consistent formula of quantification theory

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suffices to instruct the machine that it is to "print out" an answer only when it is *different* from the previous answer.

[2] Note that we do not require the function f to be such that $\bar{\mathrm{P}}(x_1, \ldots, x_n) \equiv \lim_{y\to\infty} f(x_1, \ldots, x_n, y) = 0$ — however, this condition may also be satisfied as well, by replacing the given function f by f^*, where $f^*(x_1, \ldots, x_n, y) = 1$ if $f(x_1, \ldots, x_n, y) = 1$ and $f^*(x_1, \ldots, x_n, y) = 0$ otherwise.

[3] Cf. [3].

does have a model in $\Sigma_1{}^*$.[4] (This result applies only to quantification theory *without* identity.) The proof uses Theorems 1 and 2, which are the answers to Questions 1 and 2, and the Hilbert-Bernays-Kleene result[5] that every consistent formula of quantification theory has a model in $\Sigma_2 \cap \Pi_2$. In 1957 I gave an example of a consistent formula of quantification theory with no model in which all the predicates belong to $\Sigma_1 \cup \Pi_1$ (answering another question of Mostowski).[6]

§ 2. Characterization theorems.

THEOREM 1. P *is a trial and error predicate if and only if* $P \in \Sigma_2 \cap \Pi_2$.

PROOF. Suppose P is a trial and error predicate. (I shall give the proof for a one-place predicate. The case of an n-place predicate is exactly the same except that "$x_1, x_2, \ldots, x_n$" has to be put for "x" throughout.) Then by the definition (cf. § 1), there is a g.r. function f such that for every x:

$$P(x) \equiv \lim_{y \to \infty} f(x, y) = 1$$

$$\overline{P}(x) \equiv \lim_{y \to \infty} f(x, y) = 0.$$

Now we observe that since f must approach either 0 or 1,

(1) $P(x) \equiv \lim_{y \to \infty} f(x, y) = 1$ implies that

(2) $P(x) \equiv (y)(Ez)(f(x, y) \neq 1 \supset (z > y \;\&\; f(x, z) = 1))$.

Thus $P \in \Pi_2$, and by (1) we have $P \in \Sigma_2$, since the predicate "$\lim_{y \to \infty} f(x, y) = 1$" is in Σ_2.

To prove the other half of the theorem, assume

(3) $P(x) \equiv (Ea)(b)R_1(x, a, b)$

$\overline{P}(x) \equiv (Ea)(b)R_2(x, a, b)$

where R_1 and R_2 are recursive.

Let $L(x, a, c)$ mean that a is the smallest integer such that

$$[(Eb)_{<c}(\overline{R}_2(x, a, b) \;\&\; \sim(Eb)_{<c}\overline{R}_1(x, a, b)) \;.\mathrm{v}.\; (Eb)_{<c}(\overline{R}_1(x, a, b) \;\&\; \sim(Eb)_{<c}\overline{R}_2(x, a, b))] \;.\&.\; a < c$$

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[4] An expression $(Ex)(y)R$, where R is a recursive predicate, is called a Σ_2-expression here, and $(x)(Ey)R$ is called a Π_2-expression. (Cf. [1], ch. 9; Davis, however, uses "P" and "Q" where we use Σ and Π.) We use K^* to denote the closure of a class K of predicates under truth functions. In particular, $\Sigma_1{}^*$ is the smallest class of predicates containing the r.e. predicates and closed under truth-functions.

[5] Cf. [2], p. 394, Theorem 35.

[6] Cf. [4].

Define:

(4) $f(x, y) = 1 \equiv (\mathrm{E}a)(\mathrm{L}(x, a, y) \mathbin{\&} (\mathrm{E}b)_{<y}\mathrm{R}_1(x, a, b))$
$f(x, y) = 0$ otherwise.

f is general recursive, since from the definition of L we can determine whether or not there is an a such that $\mathrm{L}(x, a, y)$, once we are given x and y (there is at most one such a). Formally, it suffices to note that a must be less than y for $\mathrm{L}(x, a, y)$ to hold, and that the definition of L contains only bounded quantifiers.

Let $\mathrm{P}(x)$ be true, and let a be the least a such that for every b $\mathrm{R}_1(x, a, b)$ holds. Since $\mathrm{P}(x)$ is true, there is for every a' at least one b such that $\overline{\mathrm{R}}_2(x, a', b)$; and since a is least, there is also for every $a' < a$ at least one b such that $\overline{\mathrm{R}}_1(x, a', b)$. There will thus be $2a$ numbers $b_0, b'_0, \ldots, b_{a-1}, b'_{a-1}$ such that $\overline{\mathrm{R}}_1(x, 0, b_0), \overline{\mathrm{R}}_2(x, 0, b'_0), \ldots, \overline{\mathrm{R}}_1(x, a-1, b_{a-1}), \overline{\mathrm{R}}_2(x, a-1, b'_{a-1})$. Taking y sufficiently large so that each of these $2a$ numbers is less than y, $a < y$, we see that $\mathrm{L}(x, a', y)$ does not hold for $a' < a$. And if y is also bigger than the least b such that $\overline{\mathrm{R}}_2(x, a, b)$, then, checking the definition, we see that $\mathrm{L}(x, a, y)$ holds. Thus, if $\mathrm{P}(x)$ is true, for sufficiently large y we will have $f(x, y) = 1$; and in a similar way we can show that if $\mathrm{P}(x)$ is false, then for sufficiently large values of y we will have both $\mathrm{L}(x, a, y)$, where a is the smallest integer such that $(b)\mathrm{R}_2(x, a, b)$, and $f(x, y) = 0$. This completes the proof of the theorem.

THEOREM 2. *There exists a k such that* P *is a k-trial predicate if and only if* P *belongs to* $\Sigma_1{}^*$, *the smallest class containing the recursively enumerable predicates and closed under truth-functions.*

PROOF. Suppose P is a k-trial predicate. Then by the definition (cf. § 1) there is a g.r. function f such that

(1) $\mathrm{P}(x) \equiv \lim_{y\to\infty} f(x, y) = 1$

(2) there are *at most k* integers y, for each x, such that $f(x, y) \neq f(x, y+1)$. (Attention is confined to one-place predicates P, for simplicity.)

Now define $\mathrm{Y}_i(x)$ (for $i = 1, 2, \ldots, k$) as meaning that there are at least i integers y such that $f(x, y) \neq f(x, y+1) \mathbin{\&} f(x, a_i+1) = 1$, where a_i is the ith integer y, in order of magnitude, such that $f(x, y) \neq f(x, y+1)$; and define $\mathrm{N}_i(x)$ as meaning that there are at least i integers y such that $f(x, y) \neq f(x, y+1) \mathbin{\&} f(x, a_i+1) \neq 1$. Finally, define $\mathrm{Y}_0(x)$ as meaning that $f(x, 0) = 1$ and $\mathrm{N}_0(x)$ as meaning that $f(x, 0) \neq 1$. Then all the predicates Y_i and N_i are recursively enumerable, and we have:

$$\mathrm{P}(x) \equiv \mathrm{Y}_k(x) \vee (\mathrm{Y}_{k-1}(x) \mathbin{\&} \overline{\mathrm{N}}_k(x)) \vee (\mathrm{Y}_{k-2}(x) \mathbin{\&} \overline{\mathrm{N}}_{k-1}(x)) \vee \ldots \vee (\mathrm{Y}_0(x) \mathbin{\&} \overline{\mathrm{N}}_1(x)).$$

In proving the other half of the theorem, we will again confine attention to one-place predicates (or sets), since the n-place case introduces no additional ideas. Let $P \in \Sigma_1^*$. Then

$$P = (A_1 - B_1) \cup (A_2 - B_2) \cup \ldots \cup (A_n - B_n) \text{ for some } n$$

where the A_i and B_i are r.e. Following Kleene [2], let $T(e, x, y)$ mean that y is the number of a computation that the number x belongs to the r.e. set with gödel number e. We define $f(x, y)$ as follows (where $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are gödel numbers of $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$);

$$f(x,y) = 1 \text{ if there are } i \leqq n,\ e < y \text{ such that } (T(a_i, x, e) \,\&\, (e)_{<y}\overline{T}(b_i, x, e))$$

$$f(x,y) = 0 \text{ otherwise.}$$

Then f has the properties (1) and (2) (taking $k = 2n$), as is easily verified.

§ 3. Applications to logic.

THEOREM 3. *Every consistent formula of quantification theory without identity has a model in* Σ_1^*.

PROOF. If A contains m predicate letters P_i, each of which is at most n-place, we construct an A′ which is obviously satisfiable if and only if A is, and which has a single $n+1$-place predicate letter and m distinct individual constants by replacing $P_i(x_1, \ldots, x_r)$ $(1 \leqq r \leqq n)$ by $P(x_1, \ldots, x_r, a_i, \ldots, a_i)$ (with $n+1$ argument places).

Suppose A′ has a model in Σ_1^*. Then the predicates P_i defined as follows:

$$P_1(x_1, \ldots, x_{r_1}) =_{\mathrm{df}} P(x_1, \ldots, x_{r_1}, a_1, \ldots, a_1)$$
$$\vdots$$
$$P_m(x_1, \ldots, x_{r_m}) =_{\mathrm{df}} P(x_1, \ldots, x_{r_m}, a_m, \ldots, a_m)$$

are also in Σ_1^*. Hence it suffices to show that if A′ is consistent then A′ has a model in Σ_1^*; it will then automatically follow that if A is consistent then A has a model in Σ^*_1.

Finally, A′ has a model in Σ_1^* if and only if its existential quantification with respect to $a_1, \ldots, a_m$ has a model in Σ_1^*. Hence the theorem reduces to the following Lemma:

LEMMA. *Every consistent formula of quantification theory (without identity) with one predicate letter and no individual constants has a model in* Σ_1^*.

To prove this we start with a model in $\Sigma_2 \cap \Pi_2$ (every consistent formula has such a model, by Theorem 35 of [2], p. 394), and modify it so as to obtain a model in Σ_1^*. Accordingly, let P be the sole predicate letter in A, and let A be true when P is interpreted as standing for the predicate F, where $F \in \Sigma_2 \cap \Pi_2$. By Theorem 1, there is a general recursive function

$f(x_1, \ldots, x_n, y)$ such that

$$F(x_1, \ldots, x_n) \equiv \lim_{y\to\infty} f(x_1, \ldots, x_n, y) = 1$$

$$\overline{F}(x_1, \ldots, x_n) \equiv \lim_{y\to\infty} f(x_1, \ldots, x_n, y) = 0$$

We define sets of integers $R(i)$ as follows:[7]
if $i \neq 0$, $R(i) = \{J(b, i)\}$ where b is the smallest integer such that for all $x_1, \ldots, x_n, y$, $(y > b \;\&\; x_1, \ldots, x_n \leqq i \Rightarrow f(x_1, \ldots, x_n, y) = f(x_1, \ldots, x_n, b))$ (i.e., b is a "modulus of convergence" of f, for $x_1, \ldots, x_n \leqq i$).

We take $R(0)$ as the set of all integers not belonging to any set $R(i)$, $i \neq 0$.

It is easily proved that the sets $R(i)$ are all disjoint and non-empty. (Towards disjointness, use the fact that $J(a, b) = J(c, d)$ implies that $b = d$; and towards non-emptiness observe that for any k, $J(k, 0) \in R(0)$.) And by the definition of $R(0)$ every integer is in one of the sets $R(i)$. Thus R^{-1} maps the set of integers many-one *onto* the set of integers. A predicate of integers and its inverse image under any function mapping the set of integers onto the set of integers satisfy the same sentences of logic without identity. Hence A is true when P is interpreted as standing for the predicate G which we define as follows:

$$G(x_1, \ldots, x_n) \equiv (Ey_1, \ldots, y_n)(F(y_1, \ldots, y_n) \;\&\; x_1 \in R(y_1) \;\&\; \ldots \;\&\; x_n \in R(y_n)).$$

It only remains to prove that $G \in \Sigma_1{}^*$.

To prove that $G \in \Sigma_1{}^*$, observe that for any integer x, there are only two possibilities: $x \in R(0)$, and $x \in R(L(x))$. Hence for any n integers $x_1, \ldots, x_n$, there are just 2^n possible cases:

1) $x_1, \ldots, x_n \in R(0)$

2) $x_1, \ldots, x_{n-1} \in R(0), x_n \in R(L(x_n))$

$\vdots$

2^n) $x_1 \in R(L(x_1)), \ldots, x_n \in R(L(x_n))$

Moreover, the truth value of $G(x_1, \ldots, x_n)$ (or $G(X)$, as we shall henceforth write for short) on the assumption that any given case holds can be effectively determined: for instance, the truth value of $G(X)$ on the assumption that case 1) holds is that of $F(0, 0, \ldots, 0)$ (which we will assume given); while if, say, case $2^n - 1$) holds, the truth value of $G(X)$ is that of $F(L(x_1), \ldots, L(x_{n-1}), 0)$. In this case we simply find the *largest* of the numbers $L(x_1), \ldots, L(x_{n-1})$. Suppose it is $L(x_j)$. Then $G(X)$ is true if $f(L(x_1), \ldots, L(x_{n-1}), 0, K(x_j)) = 1$, and false otherwise. And similarly with all the other cases.

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[7] Here J, K, L are the standard pairing and inverse-pairing functions (see [1], pp. 43–45).

We can now write down a series of zeros and ones which will terminate in 1 if $G(X)$ is true and in 0 if $G(X)$ is false, as follows:

Compute the truth value of $G(X)$ according to the assumption that case 2^n) holds, and hence $x_j \in R(L(x_j))$, where $L(x_j)$ is the largest of the numbers $L(x_i)$, and put down 1 as our "first trial answer" if the value is "truth" and 0 if the value is "falsity". The first trial answer is never revised unless an integer k is generated such that $K(x_j) < k$, but for some $z_1, \ldots, z_n \leqq L(x_j)$, it is not the case that $f(z_1, \ldots, z_n, K(x_j)) = f(z_1, \ldots, z_n, k)$. If this ever happens, then $f(z_1, \ldots, z_n, y)$ is *not* equal to $f(z_1, \ldots, z_n, K(x_j))$ for all $y > K(x_j)$, $z_1, \ldots, z_n \leqq L(x_j)$, and $x_j \in R(0)$.

If we ever discover that $x_j \in R(0)$, then we pick the largest of the remaining numbers $L(x_i)$ and repeat the reasoning to arrive at our next trial answer. (If $L(x_{j'})$ is the largest of the remaining numbers $L(x_i)$, we can determine the truth value of $G(X)$ on the assumption that $x_{j'} \in R(L(x_{j'}))$, because we now know that $x_j \in R(0)$, and so it suffices to know the truth value of $F(z_1, \ldots, z_n)$ for $z_1, \ldots, z_n \leqq L(x_{j'})$ to compute that of $G(X)$.)

Proceeding in this way, we cannot change our trial answer more than n times (since, except for the trial answer corresponding to case 1), a trial answer is put down only when it is assumed that $x_i \in R(L(x_i))$ for some i; and such an assumption is either retained forever — in which case it is correct — or abandoned at some time and never subsequently reinstated).

Let the above procedure for putting down trial answers be mechanized, and program the Turing machine so that at any stage y it repeats the last number it put down, if no new trial answer is forthcoming at that stage. Let $g(x_1, \ldots, x_n, y) =$ the number put down by the machine at the yth stage. Then g satisfies the conditions listed in Theorem 2, and it follows that $G \in \Sigma_1^*$.

COROLLARY. *Every n-place predicate in* $\Sigma_2 \cap \Pi_2$ *has an n-trial predicate as an inverse image under a suitable function mapping the integers onto the integers.* (This is what was really proved after the Lemma. Theorem 3 is an immediate consequence of this fact by the discussion preceding the Lemma. By way of contrast, recall that there exist consistent formulas with one predicate letter with no model in $\Sigma_1 \cup \Pi_1$. Thus it is *not* true that every n-place predicate in $\Sigma_2 \cap \Pi_2$ has an inverse image in $\Sigma_1 \cup \Pi_1$, even if we allow arbitrary functions mapping the integers onto the integers.)

Hitherto we have considered models in which the domain (the range of the individual variables) was the set of all non-negative integers. For models of this kind, Theorem 3 is false for predicate calculus with identity, since there are even consistent formulas with no infinite model at all. If we generalize slightly, by allowing the domain to be any recursive set, then the question whether Theorem 3 extends also to predicate calculus with identity remains open. We are, however, able to prove:

THEOREM 4. *Every consistent formula of predicate calculus with identity has a recursive model with a Π_1 domain.*

PROOF. In the foregoing proof, it suffices to modify the definition of R(0) by taking $R(0) = J(b, 0)$, where b is the smallest integer such that for all $y>b$, $f(0, \ldots, 0, y) = f(0, \ldots, 0, b)$. Let $S = \bigcup_i R(i)$. Since the mapping

$$i \leftrightarrow \text{any member of } R(i)$$

is now one-one (because $R(i)$ is now a singleton, for all i), it follows that $\langle S, G\rangle$ is a model for A in predicate calculus with identity, where G is defined as in the preceding proof. Define G^* as follows: $G^*(x_1, \ldots, x_n)$ is true if and only if the truth value of $G(x_1, \ldots, x_n)$ is "truth" *on the assumption that case* 2^n) *holds* (we showed above that this could be effectively determined). Then G^* is a recursive predicate (we make free use of Church's Thesis; however it is straightforward to eliminate it by the techniques of [2]), and G^* agrees with G whenever case 2^n) holds; hence, whenever all the arguments ϵ S. Thus $\langle S, G^*\rangle$ is also a model for A.

It remains only to show that S is a Π_1 set (i.e., S has a recursively enumerable complement). To do this, we observe that S can be defined as follows:

$$\begin{aligned} S = \{J(b,i) \mid (y)_{>b}(z_1)\ldots(z_n)(z_1,\ldots,z_n \leqq i \Rightarrow f(z_1,\ldots,z_n,y) = f(z_1,\ldots,z_n,b))\ \& \\ (b')_{<b}((z_1)\ldots(z_n)(z_1,\ldots,z_n \leqq i \Rightarrow f(z_1,\ldots,z_n,b') = f(z_1,\ldots,z_n,b)) \Rightarrow \\ (Eb'')_{<b}(Ez_1)_{\leq i}\ldots(Ez_n)_{\leq i}(f(z_1,\ldots,z_n,b') \neq f(z_1,\ldots,z_n,b'')\ \&\ b'<b''))\}. \end{aligned}$$

(To verify this, note that if b is the smallest modulus of convergence of f, for arguments bounded by i, then for every b' smaller than b and such that $f(z_1, \ldots, z_n, b) = f(z_1, \ldots, z_n, b')$ when the z's are bounded by i, there must be a b'' "between" b and b' on which $f(z_1, \ldots, z_n, b'')$ fails to equal $f(z_1, \ldots, z_n, b')$ for some z's bounded by i; for otherwise b' would already be a modulus of convergence for arguments bounded by i, and smaller than the least modulus of convergence, which is a contradiction.)

It is well known that $\Sigma_2 \cap \Pi_2$ is the class of predicates of degree of unsolvability $\leqq \mathbf{0}'$ (i.e., the class of predicates Turing reducible to K, where K is any complete r.e. set). Thus our Theorem 1 states that the trial and error predicates are exactly the ones of degree $\leqq \mathbf{0}'$. This fact makes it very easy to give informal proofs of the existence of predicates of degree $\leqq \mathbf{0}'$ satisfying various conditions, just as Church's Thesis makes it easy to give informal proofs of the existence of recursive and r.e. predicates satisfying various conditions.

** The following theorem (unpublished) is due to Hartley Rogers, Jr.:

** Added August 24, 1964.

THEOREM. *For any set A, let* **B**(A) *be the smallest class containing all sets many-one reducible to* A, *and closed under truth functions. Then* **B**(A) *contains exactly those sets which are reducible to* A *by bounded truth tables.*

Thus our Σ_1^* is the class of all sets reducible to K by bounded truth tables (where K is, again, any complete r.e. set). Hartley Rogers, Jr. has noted, after reading the present paper, that our proof of Theorem 3 can be modified to yield a construction which shows that for any A, if A is Turing reducible to K, then there exists an f and a C such that $f(\mathrm{C}) = \mathrm{A}$, $f(\bar{\mathrm{C}}) = \bar{\mathrm{A}}$, and C is reducible to K by bounded truth tables. Thus our logical result can be obtained in a way which uses only the notions of Post's 1944 paper, without the introduction of trial and error predicates and k-trial predicates. It seems to us, however, that the "modulus of convergence" idea, used in the proof of Theorem 3, is very easily understood when presented, as here, in terms of the ideas of *trial and error* and *k-trial*.

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