Restrictions on the learner

Nina Gierasimczuk   Dick de Jongh

February 15th, 2013

Definition 1 *A learning function M is* recursive *if M is a recursive function.*

We also refer to recursive identifiability as *effective* identifiability, etc.

Definition 2 *A learning function M is* consistent *if, for each σ, $content(\sigma) \subseteq L_{M(\sigma)}$.*

A consistent learner is one who never makes conjectures that contradict the input. That is not as self-evident a requirement as it might seem. In the first place it may be undecidable whether an input belongs to a certain language, which makes it of course difficult to be consistent. But there may be other reasons as well not to be consistent, in practice the learner can concentrate e.g. on simple inputs and ignore complicated ones in a case in which simple ones are sufficient for identification.

Theorem 1 *If $\mathcal{L}$ is identifiable, then $\mathcal{L}$ is identifiable by a consistent learner.*

Proof This follows from the proof of the characterization theorem: the learner constructed in the proof of theorem is not directly consistent but can be made so by changing

$M(\sigma) = \mu e'(e'$ is a number for some $L' \in \mathcal{L}$ such that
$D_{L'} \subseteq content(\sigma) \subseteq L')$ if such e' exists, and 0 otherwise.

into

$M(\sigma) = \mu e'(e'$ is a number for some $L' \in \mathcal{L}$ such that
$D_{L'} \subseteq content(\sigma) \subseteq L')$ if such e' exists, and
$\mu e'(e'$ is a number for some $L' \in \mathcal{L}$ such that
$content(\sigma) \subseteq L')$ otherwise. □

On the other hand, the combination of recursivity and consistency is *restrictive.*

Theorem 2 *If M is a consistent, recursive learner and M identifies L, then L is recursive.*

Proof Let M be consistent and recursive learner and identify L. Suppose σ is a locking sequence for M and L. Consider $M(\sigma^\wedge\langle x\rangle)$ for arbitrary x: $M(\sigma^\wedge\langle x\rangle) = M(\sigma)$ if $x \in L$ since σ is locking and $M(\sigma^\wedge\langle x\rangle) \neq M(\sigma)$ if $x \notin L$ since M is consistent. Then L is recursive because M is. $\square$

Definition 3 *A learning function M is* conservative *if, for each σ and x, content$(\sigma^\wedge\langle x\rangle) \subseteq L_{M(\sigma)}$ implies $M(\sigma^\wedge\langle x\rangle) = M(\sigma)$.*

Again it can be seen immediately that conservativity by itself is not restrictive, the learner defined in the proof of the characterization theorem is conservative. Later we will see that the combination with recursivity is restrictive.

An incremental learner (also called *memory-limited*) is one who does not act on the sequence of all previous inputs but only on the last one. This is a very efficient way of learning in practice. The learner is allowed to use its own last conjecture.

Definition 4 *A learner M is called* incremental *if, for all σ, τ, x, if $M(\sigma) = M(\tau)$, then $M(\sigma^\wedge\langle x\rangle) = M(\tau^\wedge\langle x\rangle)$.*

Theorem 3 *There exits an identifiable collection of languages that is not identifiable by an incremental learner.*

Proof Take L to be the collection containing

1. The set E of all even numbers,
2. The languages $L_n = E \cup \{2n+1\}$ for each n,
3. The languages $L'_n = E \cup \{2n+1\} - \{2n\}$ for each n.

The collection $\mathcal{L}$ is identifiable. Let us give an informal description of the learner M. M conjectures E until a number $2n+1$ is encountered. Then M switches to L'_n, except when $2n$ is part of the input already (memory!), in which case M switches to L_n. Later M switches from L'_n to L_n when $2n$ is encountered as well.

If M is an incremental learner this fails however, M will not be able to "remember" whether $2n$ has shown up or not. Let σ be a locking sequence for such an M and E, and take some $2n \notin$ *content*(σ) (there are of course infinitely many of those). Then $M(\sigma^\wedge\langle 2n\rangle) = M(\sigma)$ because $2n$ is even. The memory-limitedness of M then implies $M(\sigma^\wedge\langle 2n\rangle^\wedge\langle 2n+1\rangle) = M(\sigma^\wedge\langle 2n+1\rangle)$. But then M will treat the two texts $\sigma^\wedge\langle 2n\rangle^\wedge\langle 2n+1\rangle, 0, 2, 4, \ldots, 2n-2, 2n+2, \ldots$ and $\sigma^\wedge\langle 2n+1\rangle, 0, 2, 4, \ldots, 2n-2, 2n+2, \ldots$ in the same way although they are texts for the different languages L_n and L'_n, contradicting the assumption that M identifies $\mathcal{L}$. $\square$

It is interesting that the limitations of incrementality can be overcome by requiring more informative text.

Definition 5 *A text t for L is called a* fat text *for L if each member of L occurs infinitely many times in t.*

Example 4 *The $\mathcal{L}$ of the proof of Theorem 3 can be identified by an incremental learner on fat text: if only fat text is presented to M, M identifies $\mathcal{L}$.*

Proof Here is a description of an incremental learner M that identifies $\mathcal{L}$. M conjectures E until a number $2n+1$ is encountered. Then M switches to L'_n Later, M switches from L'_n to L_n if $2n$ is encountered as well. The difference with the successful learner of the previous proof is that, if the text t is fat, M need not "remember" that $2n$ has occurred before, if it has it will occur again. $\square$

Actually, the shortcomings of incremental learners can in all cases be overcome by fat text (see Osherson et al, p. 111), but that is rather complicated. To show the restrictiveness of effective identification we turn to r.e. sets because the examples there are simpler.

Theorem 5 *There exists a (recursively enumerable) family of sets $\mathcal{L}$ which is identifiable but not by a recursive function, namely the family $\mathcal{L} = \{K \cup \{x\} \mid x \in \mathbb{N}\}$.*

Before giving the proof we first need a recursion-theoretic lemma.

Lemma 6 *If X is an r.e. set and $\overline{X} = \{x \mid \exists y\, Bxy\}$ with B recursive, then X is recursive.*

Proof If $\overline{X} = \{x \mid \exists y\, Bxy\}$, then $\overline{X}$ is r.e., and if both X and $\overline{X}$ are r.e., then, by Post's theorem, X is recursive. $\square$

Proof of Theorem 5. Let M be recursive and identify $\mathcal{L}$. By the fact that K is r.e. but not recursive and the lemma above, it is sufficient to show that $\overline{K} = \{x \mid \exists y\, Bxy\}$ with B recursive, to obtain a contradiction. Let K be enumerated effectively as $k_1, k_2, k_3, \ldots$ and let σ be a locking sequence for M and K. Consider the texts $t_x = \sigma^\wedge\langle x\rangle^\wedge k_1, k_2, k_3, \ldots$. Note that, for each x, t_x is a text for $K \cup \{x\} = L_x$. So, we have:

If $x \in K$, then $K \cup \{x\} = K$. So, since σ is a locking sequence for M and K, for each n, $M(\sigma^\wedge\langle x, k_1, k_2, \ldots, k_n\rangle) = M(\sigma)$.

If $x \notin K$, then $K \cup \{x\} = L_x \neq K$. So, since M identifies L_x, for some n, $M(\sigma^\wedge\langle x, k_1, k_2, \ldots, k_n\rangle) \neq M(\sigma)$.

Thus, $x \in K \iff \exists y(M(\sigma^\wedge\langle x, k_1, k_2, \ldots, k_y\rangle) \neq M(\sigma))$, we have found the required Bxy as $M(\sigma^\wedge\langle x, k_1, k_2, \ldots, k_y\rangle) \neq M(\sigma)$. $\square$

Theorem 7 *There exists an $\mathcal{L}$ such that $\mathcal{L}$ is identifiable effectively and by an incremental learner, but not by an incremental, recursive learner.*

Proof Take $\mathcal{L}$ to consist of the following languages

1. $2K = \{2n \mid n \in K\}$,
2. $L_n = 2K \cup \{2n+1\}$ for each n,
3. $L'_n = 2K \cup \{2n+1\} \cup \{2n\}$ for each n.

To begin with, $\mathcal{L}$ is identified by incremental learner M_1. Learner M_1 proceeds informally as follows: it conjectures $2K$ until some $2n+1$ is encountered, or some $2n$ for some $n \notin K$ (the latter is a non-effective decision!).

In the first case, M_1 switches to L_n until $2n$ is encountered; then it switches to L'_n, which can be done incrementally, because the last conjecture was L_n.

In the second case, the language can only be L'_n and M_1 switches to that conjecture and sticks to it.

Also, $\mathcal{L}$ is identified by recursive learner M_2. Learner M_2 proceeds by conjecturing $2K$ until some $2n+1$ is encountered. Then it switches to L_n, except when $2n$ has been seen already, then it switches to L'_n (non-incrementally), and it switches to L'_n later as well when it sees $2n$.

But $\mathcal{L}$ cannot be identified by an incremental, effective learner M. Suppose M does this, and σ is a locking sequence for M and $2K$. For all $2n \in 2K$, $M(\sigma^\wedge\langle 2n\rangle) = M(\sigma)$. Note that the predicate $M(\sigma^\wedge\langle 2n\rangle) = M(\sigma)$ is recursive, but $2n \in 2K$ is not. Therefore, these two predicates cannot be co-extensive: there have to exist $2n \notin 2K$ for which also $M(\sigma^\wedge\langle 2n\rangle) = M(\sigma)$. Pick one (we do this, not M!). Since M is incremental, $M(\sigma^\wedge\langle 2n\rangle^\wedge\langle 2n+1\rangle) = M(\sigma^\wedge\langle 2n+1\rangle)$. Let s be a text for $2K$. Then M will treat the texts $\sigma^\wedge\langle 2n\rangle^\wedge\langle 2n+1\rangle^\wedge s$ and $\sigma^\wedge\langle 2n+1\rangle^\wedge s$ in the same way, but those are texts for the distinct languages L'_n and L_n. $\square$