First-order framework of inquiry

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Sources

Martin, E., and Osherson, D. (1997). Scientific Discovery Based on Belief Revision, The Journal of Symbolic Logic, Vol. 62, No. 4, pp. 1352-1370.

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Scientific Strategy

Scientific strategy = a class of scientists

Definition

A strategy is canonical for a class C of problems just in case every solvable problem in C is solved by some scientist in this strategy.

Is a strategy reliable enough?

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Is a class of scientists it canonical for a class C of (interesting) problems?

Outline

1. First-order Framework of Inquiry

2. Inquiry via Belief Revision

First-order Paradigm: Language I

To obtain the set of formulas $\mathcal{L}_{form}$, we fix:

Sym — a countable, decidable set of predicates and function symbols.

Var — a countably infinite set of variables.

First-order Paradigm: Language II

Further notation:

$Var = \{v_i \mid i \in \mathbb{N}\}$.

$\mathcal{L}_{sen} \subseteq \mathcal{L}_{form}$ — the set of sentences (no free variables).

$\mathcal{L}_{basic} \subseteq \mathcal{L}_{form}$ — the set of atomic formulas and the negations thereof.

if $\varphi \in \mathcal{L}_{form}$, then $Var(\varphi)$ is the set of free variables in φ.

$\exists$-formula is any formula equivalent to a formula in prenex normal form whose quantifier prefix is limited to existentials. Similarly for $\exists\forall$, etc.

First-order Paradigm: Structures

- Countable (finite or denumerable) structures.
- Structure $\mathcal{S}$ is a model of a set of formulas $\Gamma \subseteq \mathcal{L}_{form}$ iff there is an assignment $h : Var \rightarrow |\mathcal{S}|$, with $\mathcal{S} \models \Gamma[h]$.
- The class of models of $\Gamma \subseteq \mathcal{L}_{form}$ is denoted $MOD(\Gamma)$.

First-order Paradigm: Components

- Worlds.
- Problems.
- Environments.
- Scientists.
- Success.

Worlds

All countable structures that interpret **Sym**.

Problems

A proposition is a non-empty class of structures.

A problem is a collection of disjoint propositions.

Example

Assume **Sym** contains only a single binary predicate. Let:
P_0 be a collection of strict total orders with a least point, and
P_1 be a collection of strict total orders without a least point.

Then $\mathbf{P} = \{P_0, P_1\}$ is a problem.

Environment

Sym is *observational*.

So is the domain: the elements are given temporary names.

Definition

Given structure $\mathcal{S}$, a full assignment to $\mathcal{S}$ is any mapping of *Var* onto $|\mathcal{S}|$.

Definition

Let structure $\mathcal{S}$ and a full assignment h to $\mathcal{S}$ be given.

1. An environment for $\mathcal{S}$ and h is a sequence e such that $range(e) = \{\beta \in \mathcal{L}_{basic} \mid \mathcal{S} \models \beta[h]\}$.
2. An environment for $\mathcal{S}$ is an environment for $\mathcal{S}$ and h, for some full assignment h to $\mathcal{S}$.
3. An environment is an environment for some structure.
4. An environment for proposition P is an environment for some $\mathcal{S} \in P$.
5. An environment for problem $\mathbf{P}$ is an environment for some $P \in \mathbf{P}$.

Environments: Examples

Suppose **Sym** $= \{R\}$, structure $|\mathcal{S}| = \mathbb{N}$, R is in fact $<$.

Example

h is a full assignment to $\mathcal{S}$ such that $\{(v_i, i) \mid i \in \mathbb{N}\}$. Then one environment for $\mathcal{S}$ and h looks like this:

$$v_3 \neq v_4,\ \neg R v_0 v_0,\ R v_1 v_9,\ v_{11} = v_{11},\ v_0 \neq v_3, \ldots$$

Example

g is a full assignment to $\mathcal{S}$ such that $\{(v_{2i}, i), (v_{2i+1}, i) \mid i \in \mathbb{N}\}$. Then one environment for $\mathcal{S}$ and h looks like this:

$$v_2 = v_3,\ \neg R v_4 v_5,\ R v_1 v_9,\ v_{11} = v_{11},\ v_0 \neq v_3, \ldots$$

Environments and Structure Isomorphism

Lemma

Let two structures $\mathcal{S}$ and $\mathcal{T}$ be given.

1. *if $\mathcal{S}$ and $\mathcal{T}$ are isomorphic then the set of environments for $\mathcal{S}$ is identical to the set of environments for $\mathcal{T}$.*
2. *if some environment is both for $\mathcal{S}$ and $\mathcal{T}$ then $\mathcal{S}$ and $\mathcal{T}$ are isomorphic.*

Environments: Notation

- Take environment e and $k \in \mathbb{N}$. Then:
  - e_k is k-th element of e, and
  - $e[k]$ is the initial segment of e of length $k+1$.
- SEQ denotes the collection of proper initial segments of any environment.
- Let $\sigma \in SEQ$, if σ is non-void, then $\bigwedge \sigma$ is the conjunction of the formulas in $range(\sigma)$; if σ is void, then $\bigwedge \sigma$ is $\forall v_0(v_0 = v_0)$.
- $Var(\sigma)$ is the set of all free variables in σ.
- Given a proposition P and $\sigma \in SEQ$, we say that σ is for P just in case $\bigwedge \sigma$ is satisfiable in some member of P (similarly for $\mathbf{P}$).

Scientists

A scientist Ψ is a partial or total mapping from *SEQ* into classes of structures.

If scientist Ψ is defined on $\sigma \in SEQ$, then $\Psi(\sigma)$ is a collection of structures, thus a proposition.

Success

Definition

Let scientist Ψ be given.

1. Let environment e for proposition P be given. We say that Ψ solves P in e just in case for cofinitely many k, $\emptyset \neq \Psi(e|k) \subseteq P$. We say that Ψ solves P just in case Ψ solves P in every environment for P.
2. Let problem $\mathbf{P}$ be given. We say that Ψ solves $\mathbf{P}$ just in case Ψ solves every member of $\mathbf{P}$. In this case we say that $\mathbf{P}$ is solvable.

Solvability: Examples

Example

Sym=$\{$H$\}$, where H is a unary predicate. Given $n \in \mathbb{N}$, let P_n be the class of all structures $\mathcal{S}$ such that $card(H^{\mathcal{S}}) = n$.
$\mathbf{P} = \{P_n \mid n \in \mathbb{N}\}$ is solvable.

Example

Sym=$\{$R$\}$, where R is a binary predicate. Set
$P_y = \{\langle \mathbb{N}, \prec \rangle \mid \prec$ is isomorphic to $\omega\}$,
$P_n = \{\langle \mathbb{N}, \prec \rangle \mid \prec$ is isomorphic to $\omega^*\}$.
$\mathbf{P} = \{P_y, P_n\}$ is solvable.

First step towards characterization: locking pairs

Locking pairs:

Definition

Let scientist Ψ, proposition P, $\mathcal{S} \in P$, $\sigma \in SEQ$, and finite assignment $a : Var \rightarrow |\mathcal{S}|$ be given. We say that (σ, a) is a *locking pair* for Ψ, $\mathcal{S}$ and P just in case the following conditions hold.

1. $domain(a) \subseteq Var(\sigma)$
2. $\mathcal{S} \models \bigwedge \sigma[a]$
3. For every $\tau \in SEQ$, if $\mathcal{S} \models \exists \bar{x} \bigwedge (\sigma * \tau)[a]$, where $\bar{x}$ contains the variables in $Var(\tau) - domain(a)$, then $\emptyset \neq \Psi(\sigma * \tau) \subseteq P$.

Lemma

Let scientist Ψ, proposition P, and $\mathcal{S} \in P$ be given. Suppose that scientist Ψ solves P in every environment for $\mathcal{S}$. Then there is a locking pair for Ψ, $\mathcal{S}$, and P.

Characterization: tip-offs

Definition

A $\pi - set$ is any collection of $\forall$ formulas all of whose free variables are drawn from the same finite set.

Definition

Let problem $\mathbf{P}$ and $P \in \mathbf{P}$ be given. A tip-off for $P \in \mathbf{P}$ is a countable collection $\mathbf{t}$ of π-sets such that:

1. for every $\mathcal{S} \in P$ and full assignment h to $\mathcal{S}$, there is $\pi \in \mathbf{t}$ with $\mathcal{S} \models \pi[h]$;
2. for all $\mathcal{U} \in P' \in \mathbf{P}$ with $P' \neq P$, all full assignments g to $\mathcal{U}$, and all $\pi \in \mathbf{t}$, $\mathcal{U} \not\models \pi[g]$.

If every member of $\mathbf{P}$ has a tip-off in $\mathbf{P}$, then we say that $\mathbf{P}$ has tip-offs.

Characterization

Proposition

If problem **P** *is countable and has tip-offs, then* **P** *is solvable.*

Proposition

Every solvable problem has tip-offs.

Outline

1. First-order Framework of Inquiry
2. Inquiry via Belief Revision

Introduction

How to choose an interesting strategy?
Let's look at some rationality postulates.
Inquiry within the first-order paradigm as a process of rational belief revision in the light of data, starting from a background theory.
The scientist starts her inquiry with a set of formulas X - it represents her provisional beliefs prior to inquiry. The new data σ modifies X according to a iced scheme of belief revision, resulting in $X \dot{+} \sigma$.
The idea here is similar as in AGM.
We start off with $X \subset \mathcal{L}_{form}$ as the state of belief, without assuming X to be deductively closed.

Contraction

Definition

Let $\varphi \in \mathcal{L}_{form}$ and $B \subseteq \mathcal{L}_{form}$ be given. By a maximal subset of B that fails to imply φ is meant any subset B' of B with the following properties:

1. $B' \not\models \varphi$;
2. there is no X with $B' \subset X \subseteq B$ and $X \not\models \varphi$.

The class of all maximal subsets of B that fail to imply φ is denoted by $B \perp \varphi$. In particular, if $\models \varphi$ then $B \perp \varphi = \emptyset$, and $\bigcap(B \perp \varphi) = B$.

Lemma

$\bigcap(B \perp \varphi) = \{\psi \in B \mid (\forall D \subseteq B)(\text{ if } D \cup \{\psi\} \models \varphi \text{ then } D \models \varphi)\}$.

Definition

A mapping $\dot{-}$ from $\mathcal{P}(\mathcal{L}_{form}) \times \mathcal{L}_{form}$ to $\mathcal{P}(\mathcal{L}_{form})$ is a contraction function just in case for all $B \subseteq \mathcal{L}_{form}$ and $\varphi \in \mathcal{L}_{form}$:

1. $\bigcap(B \perp \varphi) \subseteq B \dot{-} \varphi \subseteq B$;
2. if $\not\models \varphi$ then $B \dot{-} \varphi \not\models \varphi$.

Special Kinds of Contraction

Definition

A contraction function $\dot{-}$ is stringent just in case there is a strict total ordering $\prec$ of $\mathcal{P}(\mathcal{L}_{form})$ such that for all $B \subseteq \mathcal{L}_{form}$ and invalid $\varphi \in \mathcal{L}_{form}$, $B \dot{-} \varphi$ is the $\prec$-least subset of B that does not imply φ.

Proposition

Every stringent contraction function is maxichoice.

Revision Defined from Contraction

Definition

A mapping $\dot{+}$ from $\mathcal{P}(\mathcal{L}_{form}) \times SEQ$ to $\mathcal{P}(\mathcal{L}_{form})$ is revision function just in case there is a contraction function $\dot{-}$ such that for all $B \subseteq \mathcal{L}_{form}$ and $\sigma \in SEQ$,

$$B \dot{+} \sigma = \begin{cases} B & \text{if } \sigma = \emptyset \\ (B \dot{-} \neg \bigwedge \sigma) \cup range(\sigma) & otherwise \end{cases}$$

Revision Defined from Contraction

Lemma

Let revision function $\dot{+}$ be given. Then for all $B \subseteq \mathcal{L}_{form}$ and $\sigma \in SEQ$:

1. $B \dot{+} \sigma \models \bigwedge \sigma$.
2. $B \dot{+} \sigma \subseteq B \cup range(\sigma)$.
3. *If* $B \not\models \neg \bigwedge \sigma$ *then* $B \dot{+} \sigma = B \cup range(\sigma)$.
4. *If* σ *is non void then* $B \dot{+} \sigma$ *is consistent.*

Inquiry via Revision

$$\lambda\sigma \;.\; B \dot{+} \sigma$$

Linguistic Scientists

Definition

Scientist Ψ is linguistic just in case there is $\psi : SEQ \rightarrow \mathcal{P}(\mathcal{L}_f orm)$ such that for all $\sigma \in SEQ$, $\Psi(\sigma)$ is defined iff $\psi(\sigma)$ is defined, and when both are defined $\Psi(\sigma) = MOD(\psi(\sigma))$. In this case, we say that ψ underlies Ψ.

Scientists Based on Revision

Definition

Let revision function $\dot{+}$ be given. Then $\lambda\sigma.B\dot{+}\sigma$ is a linguistic scientist, which we qualify as revision-based.

The Inductive Power of Stringent Revision

Theorem

There is a stringent revision function $\dot{+}$ with the following property. Let problem $\mathbf{P}$ *be such that for some* $Y \subseteq \mathcal{L}_{form}$ *and revision function* $\dot{\oplus}$, $\lambda\sigma.Y\dot{\oplus}\sigma$ *solves* $\mathbf{P}$. *Then there is a consistent* $X \subseteq \mathcal{L}_{form}$ *such that* $\lambda\sigma.Y\dot{+}\sigma$ *solves* $\mathbf{P}$.