Rational Belief: Four Approaches, One Theory

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The theory combines doxastic logic, AGM belief revision & probability theory.

Plan:

1. The Humean Thesis on Belief
2. The "$\leftarrow$" of the Lockean Thesis and Doxastic Logic
3. The "$\rightarrow$" of the Lockean Thesis and Belief Revision Theory
4. The Closest Qualitative Approximation of Probability
5. Conclusions

(We will not have time to discuss a lot of literature—maybe in the Q&A?)

. . . an opinion or belief is nothing but an idea, that is different from a fiction, not in the nature or the order of its parts, but in the manner of its being conceived. . . An idea assented to feels different from a fictitious idea, that the fancy alone presents to us: And this different feeling I endeavour to explain by calling it a superior force, or vivacity, or solidity, or firmness, or steadiness. [. . .] its true and proper name is belief, which is a term that every one sufficiently understands in common life. [. . .] It gives them [the ideas of the judgment] more force and influence; makes them appear of greater importance; infixes them in the mind; and renders them the governing principles of all our actions. (Treatise, Section VII, Part III, Book I)

. . . the mind has a firmer hold, or more steady conception of what it takes to be matter of fact, than of fictions. (Treatise, Appendix)

Tradition in Hume interpretation has it that beliefs are lively ideas. In my interpretation, beliefs are steady dispositions. (Loeb 2002)

Hume maintains that stability is the natural function of belief. (Loeb 2002)

A disposition to vivacity is a disposition to experience vivacious ideas, ideas that possess the degree of vivacity required for occurrent belief. Some dispositions to vivacity are unstable in that they have a tendency to change abruptly... Such dispositions, in Hume's terminology, lack fixity. Hume in effect stipulates that a dispositional belief is an infixed disposition to vivacity... (Loeb 2010)

Now we are going to explicate this:

- (Rational) Degree of vivacity $\approx$ subjective probability P (cf. Maher 1981)
- Let us call then the following the *Humean thesis on rational belief:*

  It is rational to believe a proposition just in case it is rational to have a stably high degree of belief in it.

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The Humean Thesis Explicated

If Bel is a perfectly rational agent's class of believed propositions at a time, and if P is the same agent's subjective probability measure at the same time, then

$$Bel(X) \text{ iff for all } Y, \text{ if } Y \in \mathcal{Y} \text{ and } P(Y) > 0, \text{ then } P(X|Y) > r.$$

But stably high probability with respect to what class $\mathcal{Y}$ of propositions?

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The Humean Thesis Explicated

If Bel is a perfectly rational agent's class of believed propositions at a time, and if P is the same agent's subjective probability measure at the same time, then

$$(HT^r)\ \ Bel(X) \text{ iff for all } Y,\ \text{if } Poss(Y) \text{ and } P(Y) > 0, \text{ then } P(X|Y) > r$$

where $Poss(X)$ *iff not* $Bel(W \setminus X)$ *(and* $\frac{1}{2} \leq r < 1$*).*

First observation:

Doxastic logic follows from the Humean thesis (and probability theory).

Theorem

If P is a probability measure, if Bel satisfies the Humean thesis HT^r, and if not $Bel(\varnothing)$, then the principles of doxastic logic hold:

- $Bel(W)$.
- *If* $Bel(X)$ *and* $X \subseteq Y$, *then* $Bel(Y)$.
- *If* $Bel(X)$ *and* $Bel(Y)$, *then* $Bel(X \cap Y)$.
- *If* $Bel(X)$ *then* $Poss(X)$.

It follows also that there must be a least believed proposition B_W that is non-empty and which generates our perfectly rational agent's belief system Bel.

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Secondly, belief can be represented in purely probabilistic terms:

Representation theorem

The following two statements are equivalent:

I. *P is a prob. measure, Bel satisfies the Humean thesis HT^r.*

II. *P is a prob. measure, and there is a (uniquely determined) proposition X, s.t.*

- *X is a non-empty P-stable^r proposition,*
- *if $P(X) = 1$ then X is the least proposition with probability 1; and:*

For all Y:

$$Bel(Y) \text{ if and only if } Y \supseteq X$$

(and hence, $B_W = X$).

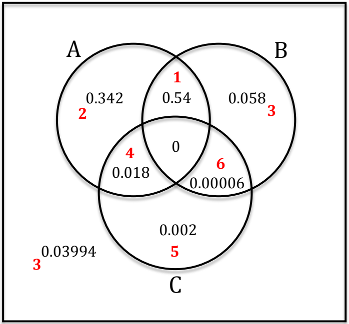
Definition (cf. Skyrms 1977, 1980; also: *strong belief*)

Let P be a probability measure. For all propositions X:

X is P-stable^r iff for all Y with $Y \cap X \neq \varnothing$ and $P(Y) > 0$: $P(X|Y) > r$.

Example:

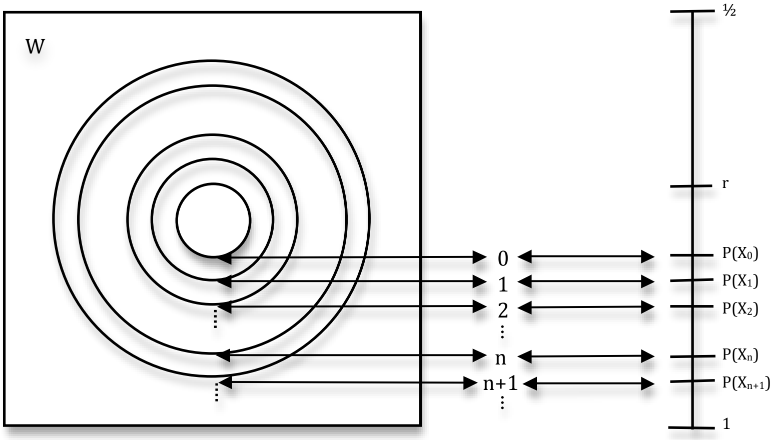
6. $P(\{w_7\}) = 0.00006$ ("Ranks")
5. $P(\{w_6\}) = 0.002$
4. $P(\{w_5\}) = 0.018$
3. $P(\{w_3\}) = 0.058$, $P(\{w_4\}) = 0.03994$
2. $P(\{w_2\}) = 0.342$
1. $P(\{w_1\}) = 0.54$

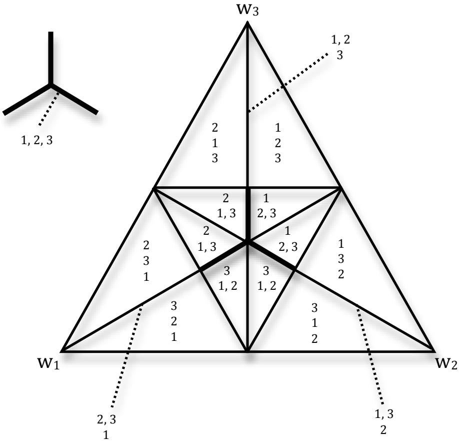


This yields the following P-stable$^{\frac{1}{2}}$ sets (B_W):

- $\{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$
- $\{w_1, w_2, w_3, w_4, w_5, w_6\}$
- $\{w_1, w_2, w_3, w_4, w_5\}$
- $\{w_1, w_2, w_3, w_4\}$
- $\{w_1, w_2\}$
- $\{w_1\}$ ("Spheres")

One can prove: The class of P-stable^r propositions X with $P(X) < 1$ is *well-ordered* with respect to the subset relation (it is a "sphere system").





Almost all P allow for a P-stable$^{\frac{1}{2}}$ set X with $P(X) < 1$!

One can also prove that there is a simple algorithm which generates all P-stable^r sets (i.e., all possible ways of satisfying the Humean thesis).

The "$\leftarrow$" of the Lockean Thesis and Doxastic Logic

most of the Propositions we think, reason, discourse, nay act upon, are such, as we cannot have undoubted Knowledge of their Truth: yet some of them border so near upon Certainty, that we make no doubt at all about them; but assent to them firmly, and act, according to that Assent, as resolutely, as if they were infallibly demonstrated... (Locke, Book IV, Essay)

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Explication (cf. Foley 1993):

- *The Lockean thesis*: $Bel(X)$ iff $P(X) \geq r > \frac{1}{2}$.

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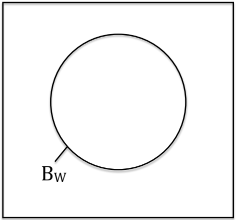
Explication (cf. Foley 1993):

- *The Lockean thesis*: $Bel(X)$ iff $P(X) \geq r > \frac{1}{2}$.

In our next approach we will combine:

- $\mathrm{LT}_{\leftarrow}^{\geq r > \frac{1}{2}}$: $Bel(X)$ if $P(X) \geq r > \frac{1}{2}$,
- the *axioms of probability* for P, and
- *doxastic logic* for Bel.



Representation theorem

The following two statements are equivalent:

I. *P is a prob. measure, Bel satisfies doxastic logic,*
the right-to-left of the Lockean thesis $\mathrm{LT}_{\leftarrow}^{\geq P(B_W) > \frac{1}{2}}$ holds.

II. *P is a prob. measure, and there is a (uniquely determined) X, s.t.*

– *X is a non-empty P-stable$^{\frac{1}{2}}$ proposition,*

– *if $P(X) = 1$ then X is the least proposition with probability 1; and:*

For all Y:

$$Bel(Y) \text{ if and only if } Y \supseteq X$$

(and hence, $B_W = X$).

And either side implies the *full* $\mathrm{LT}_{\leftrightarrow}^{\geq P(B_W) > \frac{1}{2}}$: $Bel(X)$ iff $P(X) \geq P(B_W) > \frac{1}{2}$.

(No Lottery paradox emerges from this, since the threshold co-depends on P.)

Back to the example from before:

6. $P(\{w_7\}) = 0.00006$ ("Ranks")
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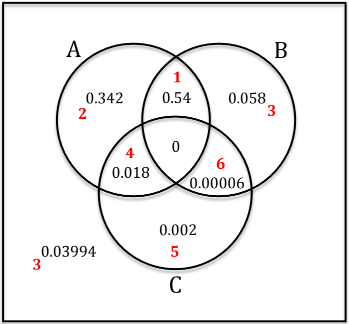

This yields the following P-stable$^{\frac{1}{2}}$ sets (B_W):

- $\{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$ (≥ 1.0)
- $\{w_1, w_2, w_3, w_4, w_5, w_6\}$ (≥ 0.99994)
- $\{w_1, w_2, w_3, w_4, w_5\}$ (≥ 0.99794)
- $\{w_1, w_2, w_3, w_4\}$ (≥ 0.97994)
- $\{w_1, w_2\}$ (≥ 0.882)
- $\{w_1\}$ (≥ 0.54) ("Spheres")

The "$\rightarrow$" of the Lockean Thesis and Belief Revision Theory

In our third approach we start from:

- *Conditional* belief $Bel(.|.)$

  (the qualitative counterpart of conditional probability).

  Read: $Bel(Y|X)$ iff the agent has *a belief in Y on the supposition of X*.

  $Bel(Y)$ iff $Bel(Y|W)$ iff the agent *believes Y (unconditionally)*.

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- Accordingly, *logical closure* of $Bel(.|.)$ needs to be redefined:

  AGM belief revision $\approx$ Lewisian conditional logic $\approx \ldots$

- "$\rightarrow$" of the Lockean Thesis (with the threshold r being *independent* of P).

Representation theorem

The following two statements are equivalent:

I. *P is a prob. measure, Bel satisfies logical closure (AGM belief revision $\approx$ Lewisian conditional logic $\approx \ldots$), and* $\mathrm{LT}^{\geq r}_{\rightarrow}$.

II. *P is a prob. measure, and there is a (uniquely determined) chain $\mathcal{X}$ of non-empty P-stable^r propositions, such that $Bel(\cdot|\cdot)$ is given by $\mathcal{X}$ in a Lewisian sphere-system-like manner.*

$\mathrm{LT}^{\geq r}_{\rightarrow}$ ("$\rightarrow$" of Lockean thesis) For all Y, s.t. $P(Y) > 0$:

For all Z, if $Bel(Z|Y)$, then $P(Z|Y) > r$.

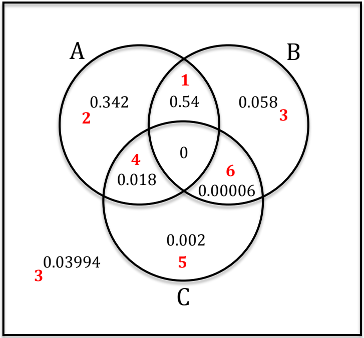
And either side implies a version of the *full* Lockean thesis again!

Example: Let P be again as in our initial example.

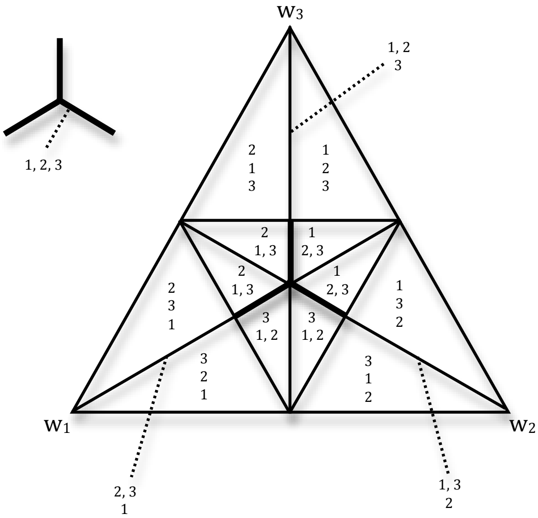
Then if $Bel(\cdot|\cdot)$ obeys AGM, and if P and $Bel(\cdot|\cdot)$ jointly satisfy $LT^{>\frac{1}{2}}_{\rightarrow}$, then $Bel(\cdot|\cdot)$ must be given by some coarse-graining of the ranking in red below.

Choosing the maximal (most fine-grained) $Bel(\cdot|\cdot)$ yields the following:

- $Bel(A \wedge B \,|\, A)$ $(A \rightarrow A \wedge B)$
- $Bel(A \wedge B \,|\, B)$ $(B \rightarrow A \wedge B)$
- $Bel(A \wedge B \,|\, A \vee B)$ $(A \vee B \rightarrow A \wedge B)$
- $Bel(A \,|\, C)$ $(C \rightarrow A)$
- $\neg Bel(B \,|\, C)$ $(C \not\rightarrow B)$
- $Bel(A \,|\, C \wedge \neg B)$ $(C \wedge \neg B \rightarrow A)$
- $\neg Bel(B \,|\, \neg A)$ $(\neg A \not\rightarrow B)$



For three worlds again (and $r = \frac{1}{2}$), the maximal $Bel(\cdot|\cdot)$ given P and r are given by these rankings (or sphere systems):



The Closest Qualitative Approximation of Probability

De Finetti and others hold that what is really important about a probability measure P is the preference order $\preccurlyeq_P$ over propositions that it determines:

$$A \preccurlyeq_P B \text{ iff } P(A) \leq P(B).$$

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But in nonmonotonic reasoning / belief revision theory we generate preferences over propositions from total pre-orders over worlds (cf. Halpern 2005):

$$max_{\preccurlyeq}(A) = \{w \in A \,|\, \forall w' \in A, w' \preccurlyeq w\}$$

$$A \preccurlyeq B \text{ iff } \forall w \in max_{\preccurlyeq}(A), \exists w' \in max_{\preccurlyeq}(B)\text{: } w \preccurlyeq w'.$$

(And from $max_{\preccurlyeq}(.)$ one can also determine absolute and conditional belief.)

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(And from $max_{\preccurlyeq}(.)$ one can also determine absolute and conditional belief.)

Question: To what extent can we approximate $\preccurlyeq_P$ by $\preccurlyeq$?

Equally importantly: What exactly do we *mean* by approximation here?

When aiming to approximate the truth by means of belief, and assuming that a rational agent's set of beliefs is consistent, there are just two kinds of errors:

- The Soundness Error: Believing A when A is false.
- The Completeness Error: Neither believing A nor $\neg A$ (when either is true).

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Replacing suspension by indifference leads us to the corresponding two kinds of errors when aiming to approximate $\preccurlyeq_P$ by $\preccurlyeq$:

- The Soundness Error: $A \prec B$ when it is not the case that $A \prec_P B$.
- The Completeness Error: $A \approx B$ when either $A \prec_P B$ or $B \prec_P A$ is true.

Hence we define:

Definition

For all total pre-orders $\preccurlyeq, \preccurlyeq'$ over W (from which total pre-orders $\preccurlyeq, \preccurlyeq'$ over propositions can be defined), and for all probability measures P:

$\preccurlyeq$ is more accurate than $\preccurlyeq'$ (when approximating $\preccurlyeq_P$) iff

1. the set of soundness errors of $\preccurlyeq$ is a proper subset of the set of soundness errors of $\preccurlyeq'$;
2. or: their sets of soundness errors are equal, but the set of completeness errors of $\preccurlyeq$ is a proper subset of the set of completeness errors of $\preccurlyeq'$.

Theorem

For all probability measures P (defined on all subsets of W):

- *For all total pre-orders $\preccurlyeq$ on W:*

  $\preccurlyeq$ is not subject to any soundness errors (relative to $\preccurlyeq_P$) iff

  $\preccurlyeq$ satisfies the following Sum Condition (cf. Benferhat et al., Snow): for all $w \in W$ with $P(\{w\}) > 0$,

  $$P(\{w\}) > \sum_{w' \colon w' \prec w} P(\{w'\});$$

  furthermore, for all $w \in W$ with $P(\{w\}) = 0$: for all $w' \in W$, $w \prec w'$.

- *For all total pre-orders $\preccurlyeq, \preccurlyeq'$ on W that are not affected by any soundness errors (relative to $\preccurlyeq_P$):*

  $\preccurlyeq$ is more accurate than $\preccurlyeq'$ (when approximating $\preccurlyeq_P$) iff

  $\preccurlyeq$ is more fine-grained than $\preccurlyeq'$.

It follows immediately from the theorem above that for all P there is a total pre-order $\preccurlyeq_{acc,P}$, such that:

> For all total pre-orders $\preccurlyeq'$ on W that are distinct from $\preccurlyeq_{acc,P}$,
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> $\preccurlyeq_{acc,P}$ is more accurate than $\preccurlyeq'$ (when approximating $\preccurlyeq_P$).

By the theorem, $\preccurlyeq_{acc,P}$ is simply *the most fine-grained $\preccurlyeq$ that satisfies the Sum Condition*.

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In terms of our example, $\preccurlyeq_{acc,P}$ is:

- $\{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$
- $\{w_1, w_2, w_3, w_4, w_5, w_6\}$
- $\{w_1, w_2, w_3, w_4, w_5\}$
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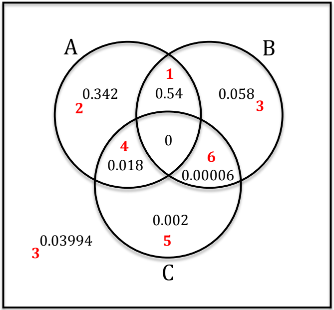

Conclusions

We have encountered four ways of justifying one and the same theory of rational belief. That theory says:

- Rational belief (*Bel*) is closed under logic and AGM belief revision.
- The rational degree-of-belief function (*P*) obeys the axioms of probability.
- Belief corresponds to resiliently high probability.

  (*Not* necessarily a reduction!)
- The theory avoids the usual paradoxes, and it has lots of applications.

  (For instance, our example measure is from a paper in Bayesian philosophy of science which we can thus give a qualitative interpretation.)

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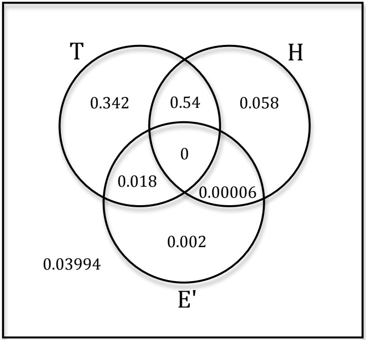
Future: Extend this to the social domain!

(Relativize P and *Bel* to worlds and agents: iterate P, *Bel*; common belief;...)

Postscript: Our Example

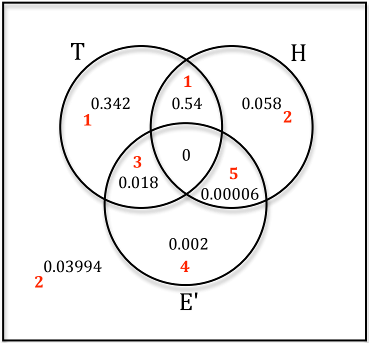
- An example in *Philosophy of Science: J. Dorling (1979)*



E': Observational result for the secular acceleration of the moon.
T: Relevant part of Newtonian mechanics.
H: Auxiliary hypothesis that tidal friction is negligible.

$P(T|E') = 0.8976$, $P(H|E') = 0.003$.

while I will insert definite numbers so as to simplify the mathematical working, nothing in my final qualitative interpretation... will depend on the precise numbers...



$$Bel(T|E'),\ Bel(\neg H|E') \ \ (\text{with } r = \tfrac{3}{4}).$$

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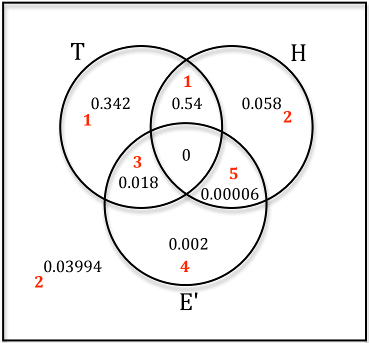

$$Bel(T|E'),\ Bel(\neg H|E')\ \ (\text{with } r = \tfrac{3}{4}).$$

... scientists always conducted their serious scientific debates in terms of finite qualitative subjective probability assignments to scientific hypotheses (Dorling 1979). ✓