Plausibility Orderings in Dynamic Games

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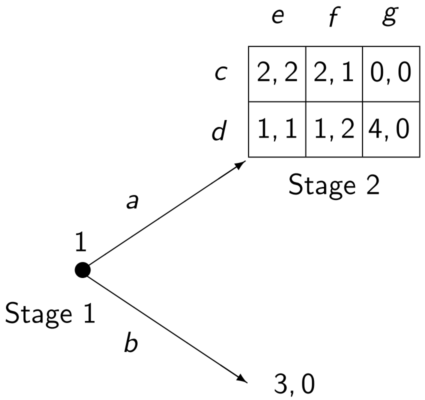
Introduction

- In dynamic games, players' **initial beliefs** about the opponents may be **contradicted** during the game.
- Players must be prepared to **revise** their beliefs about the opponents.
- **How** players **revise** their beliefs is crucial for how they **choose!**



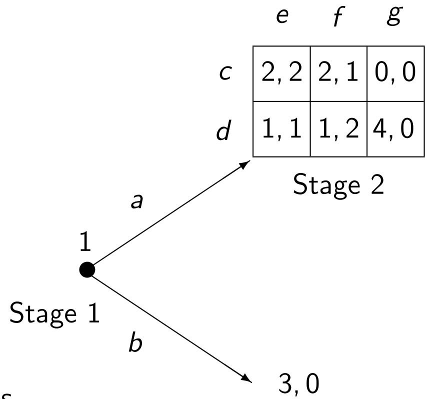
Player 2 initially believes

p : player 1 chooses rationally at stage 1

q : player 1 chooses rationally at stage 2

r : player 1 believes that player 2 chooses rationally

- Upon observing a, player 2 must withdraw at least one of these beliefs. Which?



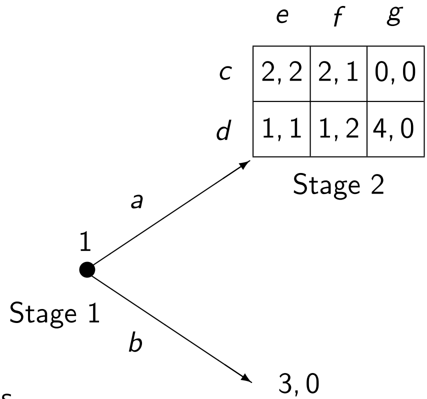
Player 2 initially believes

p : player 1 chooses rationally at stage 1

q : player 1 chooses rationally at stage 2

r : player 1 believes that player 2 chooses rationally

- **Common strong belief in rationality:** Upon observing a, player 2 withdraws belief r, but maintains p and q.
- Player 2 chooses f.



Player 2 initially believes

p : player 1 chooses rationally at stage 1

q : player 1 chooses rationally at stage 2

r : player 1 believes that player 2 chooses rationally

- **Common belief in future rationality:** Upon observing *a*, player 2 withdraws belief *p*, but maintains *q* and *r*.
- Player 2 chooses *e*.

- In this talk, we analyze "common strong belief in rationality" and "common belief in future rationality" within the framework of **belief revision theory.**
- Can these concepts be modelled by **plausibility orderings?**

Modelling Belief Hierarchies

Definition (Epistemic model)

Consider a dynamic game G with two players. An epistemic model for G is a tuple $M = (T_1, T_2, b_1, b_2)$ where

(a) T_i is a set of **types** for player i,

(b) b_i assigns to every type $t_i \in T_i$ and every information set $h \in H_i$ some **conditional belief** $b_i(t_i, h) \in \Delta(S_j(h) \times T_j)$.

M is **complete** if for every conditional belief vector $\beta_i = (\beta_i(h))_{h \in H_i}$ on $S_j \times T_j$ there is some $t_i \in T_i$ with $b_i(t_i) = \beta_i$.

- Here, $S_j(h)$ is set of strategies for player j that reach information set h.

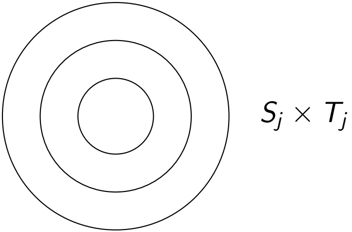
Plausibility Orderings

Definition (Plausibility ordering)

A **plausibility ordering** for player i on $S_j \times T_j$ is a reflexive and transitive relation $\preceq_i$ on $S_j \times T_j$.

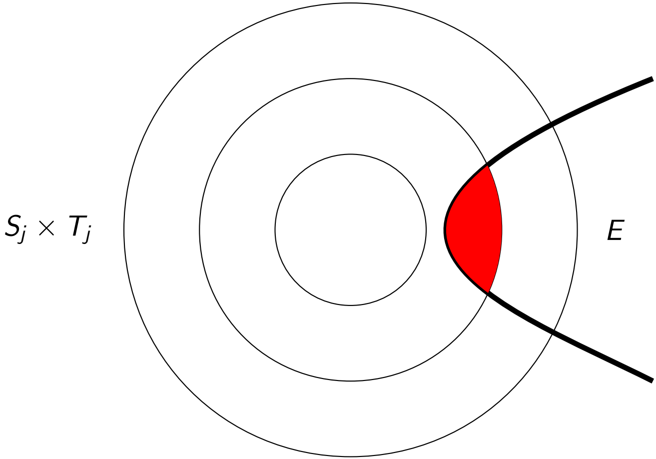
$(s_j, t_j) \prec_i (s'_j, t'_j)$ means (s_j, t_j) is deemed **more plausible** than (s'_j, t'_j).

- Corresponds to **system of spheres** in Grove (1988).



- For a given subset $E \subseteq S_j \times T_j$, define

$$\min_{\preceq_i}(E) := \{(s_j, t_j) \in E \mid \nexists\ (s_j', t_j') \in E \text{ with } (s_j', t_j') \prec_i (s_j, t_j)\}.$$



- Corresponds to **best rationalization principle** (Battigalli (1996)).

- A **reasoning concept** specifies for every dynamic game G and every **complete** epistemic model $M = (T_1, T_2, b_1, b_2)$ for G, a subset of types $\rho_i(M) \subseteq T_i$ for every player i.

Definition (Characterization by plausibility ordering)

Reasoning concept ρ is **characterized by plausibility ordering** $\preceq_i$ on $S_j \times T_j$ if for every information set $h \in H_i$:

$$\bigcup_{t_i \in \rho_i(M)} \text{supp}(b_i(t_i, h)) = \min_{\preceq_i}(S_j(h) \times T_j).$$

Strong Belief

- Consider some **event** $E \subseteq S_j \times T_j$.

Definition (Strong belief)

Type t_i **strongly believes** in E if at every information set $h \in H_i$:

$$b_i(t_i, h)(E) = 1 \text{ whenever } E \cap (S_j(h) \times T_j) \neq \emptyset.$$

Common Strong Belief in Rationality

- Strategy s_i is **rational** for type t_i if at every information set $h \in H_i$:

$$u_i(s_i, b_i(t_i, h)) \geq u_i(s_i', b_i(t_i, h)) \text{ for all } s_i' \in S_i(h).$$

- For every $\tilde{T}_j \subseteq T_j$, define

$$(S_j \times \tilde{T}_j)^{rat} := \{(s_j, t_j) \in S_j \times \tilde{T}_j \mid s_j \text{ rational for } t_j\}.$$

Definition (Battigalli and Siniscalchi (2002))

Consider a **complete** epistemic model $M = (T_1, T_2, b_1, b_2)$.

Induction start: $T_i^0 := T_i$.

Induction step: $T_i^k = \{t_i \in T_i^{k-1} \mid t_i \text{ strongly believes } (S_j \times T_j^{k-1})^{rat}\}$.
$T_i^\infty := \cap_{k \in \mathbb{N}} T_i^k$: types that express **common strong belief in rationality**.

Theorem

"Common strong belief in rationality" **can** *be characterized by a plausibility ordering.*

Proof:

- Define plausibility ordering $\preceq_i$ on $S_j \times T_j$ as follows:

  $(s_j, t_j) \prec_i (s'_j, t'_j)$ if and only if $(s_j, t_j) \in (S_j \times T_j^k)^{rat}$ and $(s'_j, t'_j) \notin (S_j \times T_j^k)^{rat}$ for some k.

- Then, at every information set $h \in H_i$:

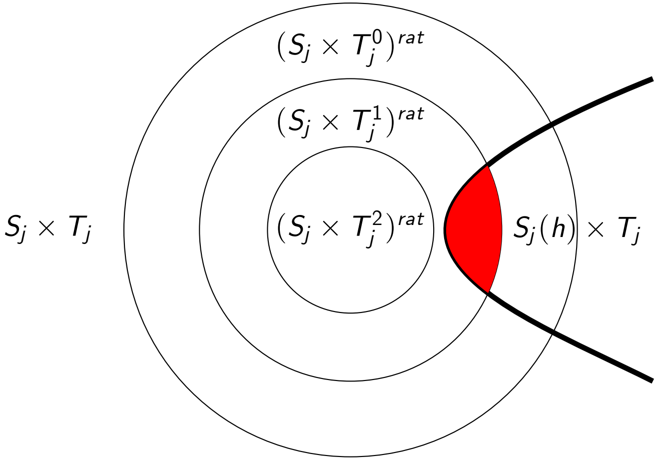
  $$\min_{\preceq_i}(S_j(h) \times T_j) = (S_j \times T_j^m)^{rat} \cap (S_j(h) \cap T_j)$$

  where m is largest k for which $(S_j \times T_j^k)^{rat} \cap (S_j(h) \cap T_j) \neq \emptyset$.

- Hence,

  $$\min_{\preceq_i}(S_j(h) \times T_j) = \bigcup_{t_i \in T_i^\infty} \mathrm{supp}(b_i(t_i, h)).$$

- Belief revision in **common strong belief in rationality.**



Belief in Future Rationality

Definition (Belief in Future Rationality)

Type t_i **believes in** j**'s future rationality** if at every information set $h \in H_i$, the conditional belief $b_i(t_i, h)$ only assigns positive probability to strategy-type pairs (s_j, t_j) where s_j is rational for t_j at every information set $h' \in H_j$ that **weakly follows** h.

- Corresponds to **stable belief in dynamic rationality** (Baltag, Smets and Zvesper (2009)).
- At $h \in H_i$, player i need not believe that j has chosen rationally in the past, **even when this is possible.**

Common Belief in Future Rationality

Definition (Common belief in future rationality)

Consider a **complete** epistemic model $M = (T_1, T_2, b_1, b_2)$.

Induction start: $T_i^1 := \{t_i \in T_i \mid t_i$ believes in j's future rationality$\}$.

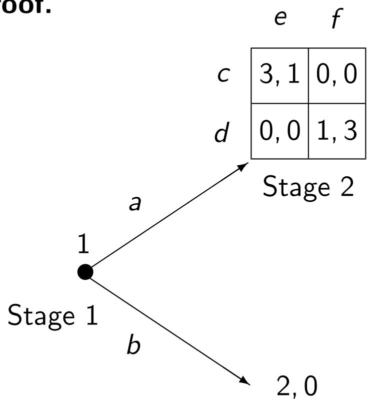
Induction step: $T_i^k = \{t_i \in T_i^{k-1} \mid b_i(t_i, h)(S_j \times T_j^{k-1}) = 1$ at every $h \in H_i\}$.

$T_i^\infty := \cap_{k \in \mathbb{N}} T_i^k$: types that express **common belief in future rationality**.

Theorem

Common belief in future rationality **cannot** *be characterized by plausibility ordering.*

Proof.



- CBFR selects (a, c) and b for player 1.
- Suppose, CBFR is characterized by plausibility ordering.
- Then, ordering must put (a, c) and b in **inner** sphere.
- Hence, at stage 2, player 2 can only deem (a, c) possible.
- This **contradicts** CBFR!

AGM Axioms

Theorem (Grove (1988))

A belief revision rule is characterized by a plausibility ordering, if and only, it satisfies the AGM axioms.

- Belief revision in **common strong belief in rationality** is compatible with AGM axioms.
- Belief revision in **common belief in future rationality** must violate some of the AGM axioms!
- Common belief in future rationality violates the **preservation axiom** in AGM theory:

Definition (Preservation axiom)

K4 If $T + p$ is consistent, then $T * p \vdash T + p$.

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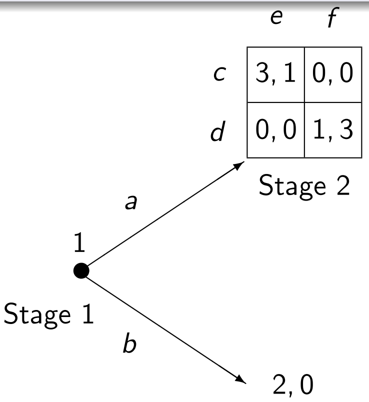

- T = player 1 chooses (a, c) or b
- p = player 1 has chosen a
- $T + p$ is consistent
- $T + p$ = player 1 chooses (a, c)
- But CBFR does not require player 2 at stage 2 to believe (a, c)!

- Preservation axiom reflects **forward induction.**