Conclusive Update and Computability

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LEARNING

DEL intuitive, determinate manners of updating models

FLT no prescribed ways of learning but restricted by computability

Compare the two aspects: determinateness and computability.

Outline

Epistemic spaces and learning

Conclusive learnability

Computational assumption on agency

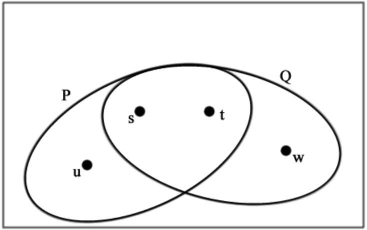
Preset learning

Fastest learning

An agent's uncertainty is represented by an epistemic space (S, Φ), where:

- $S = \{s_0, s_1, \ldots\}$ of epistemic possibilities, or possible worlds, and
- $\Phi \subseteq \mathcal{P}(S)$ a family of propositions.

Φ represent facts or observables being true or false in possible worlds.



Epistemic spaces and learning

- Learner L receives information about a possible world (the actual one).
- The information is an open-ended (infinite) sequence of propositions.
- Data stream $\varepsilon = (\varepsilon_1, \varepsilon_2 \dots)$ is a data stream for $s \in S$ just in case

$$\{\varepsilon_n : n \in \mathbb{N}\} = \{p \in \Phi : s \in p\}.$$

- Learner L is a function that on input of an epistemic space (S, Φ) and a finite sequence of observations $\sigma = (\sigma_0, \dots, \sigma_n)$ outputs a hypothesis, i.e.,

$$L((S, \Phi), \sigma) \subseteq S.$$

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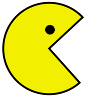

Basic types of learnability

(S, Φ)
$s_1 : p_1, p_3, p_4$
$s_2 : p_2, p_4, p_5$
$s_3 : p_1, p_3, p_5$
$s_4 : p_4, p_6$

(S', Φ')
$t_1 : p_1, p_3, p_4$
$t_2 : p_2, p_4, p_5$
$t_3 : p_1, p_3, p_5$
$t_4 : p_1, p_3, p_4, p_6$

Basic types of learnability

(S, Φ)
$s_1 : 1, 3, 4$
$s_2 : 2, 4, 5$
$s_3 : 1, 3, 5$
$s_4 : 4, 6$

(S', Φ')
$t_1 : 1, 3, 4$
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Conclusive Learnability

- Certainty in finite time.
- Only one answer,
- based on certainty.
- No chance to change later.

Conclusive learnability: once-definedness

The range of learning function L is extended by $\uparrow$ ("I do not know").

Definition

Learning function L is once defined on (S, Φ) iff for any stream ε for any world in S there is exactly one $n \in \mathbb{N}$ such that $L(\varepsilon \upharpoonright n) \in \mathbb{N}$ (i.e., is not an $\uparrow$-answer).

DEFINITION

Take an epistemic space (S, Φ).

- A world $s_m \in S$ is conclusively learnable in a computable way by a function L if L is computable, once-defined, and for every data stream ε for s, there exists a finite stage n such that $L((S, \Phi), \varepsilon_0, \ldots, \varepsilon_k) = \{m\}$.

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- The epistemic space (S, Φ) is said to be conclusively learnable in an computable way by L if L is computable and all its worlds in S are conclusively learnable in an computable way by L.
- Finally, the epistemic space (S, Φ) is conclusively learnable in an computable way just in case there is a computable learning function that can conclusively learn it in an computable way.

Definition

Take (S, Φ). A set $D_i \subseteq \Phi$ is a definite finite tell-tale set (DFTT) for s_i in S if:

1. D_i is finite,
2. $s_i \in \bigcap D_i$, and
3. for any $s_j \in S$, if $s_j \in \bigcap D_i$ then $s_i = s_j$.

Theorem

(S, Φ) is conclusively learnable in an computable way just in case there is a computable function $f : S \to \mathcal{P}^{<\omega}(\Phi)$ s.t. $f(s)$ is a DFTT for s.

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a world is conclusively learnable

if it makes true a finite conjunction of propositions

that together is false everywhere else

DEFINITION

Take (S, Φ) and $x \in \Phi$. The eliminative power of x with respect to (S, Φ) is determined by a function $El_{(S,\Phi)} : \Phi \to \mathcal{P}(\mathbb{N})$, such that:

$$El_{(S,\Phi)}(x) = \{i \,|\, s_i \notin x \;\&\; s_i \text{ in } S\}.$$

Additionally, for $X \subseteq \Phi$ we write $El_{(S,\Phi)}(X)$ for $\bigcup_{x \in X} El_{(S,\Phi)}(x)$.

eliminative power of a proposition is the complement of its extension

Definition (Fin-Id Problem)

Instance: A finite epistemic space (S, Φ), a world s_i in S.
Question: Is s_i conclusively learnable within (S, Φ)?

Theorem

Fin-Id *Problem is in P.*

Minimality of DFTT's: two kinds

set	a minimal DFTT	minimal-size DFTTs
$\{5, 7, 8\}$	$\{7, 8\}$	$\{5, 8\}$ or $\{7, 8\}$
$\{6, 8, 9\}$	$\{8, 9\}$	$\{6\}$
$\{5, 7, 9\}$	$\{7, 9\}$	$\{5, 9\}$ or $\{7, 9\}$
$\{8, 10\}$	$\{10\}$	$\{10\}$

Minimality of DFTT's: complexity

finding a minimal DFTT is easy

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Proposition

Let (S, Φ) be a conclusively learnable finite epistemic space. Finding a minimal DFTT of s_i in (S, Φ) can be done in polynomial time w.r.t. $card(\{x | s_i \in x\})$.

Minimality of DFTT's: complexity

finding a minimal-size DFTT is (most probably) harder

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Definition (Min-size DFTT Problem)

Instance: (S, Φ), $s_i \in S$, and $k \leq card(\{p|s_i \in p\})$.
Question: Is there a DFTT X_i of s_i of size $\leq k$?

Theorem

The Min-size DFTT *Problem is NP-complete.*

finding a minimal-size DFTT is (most probably) harder

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teaching efficiently might be hard

Computational assumption about epistemic spaces

Definition

An epistemic space (S, Φ) is uniformly decidable just in case there is a computable function $f : S \times \Phi \to \{0, 1\}$ such that:

$$f(s, p) = \begin{cases} 1 & \text{if } s \in p, \\ 0 & \text{if } s \notin p. \end{cases}$$

Uniform decidability and agency

- In epistemic logic uniform decidability is primitive.
- However the problem is non-trivial, e.g., in scientific scenarios.
- Epistemic space represents an uncertainty of a TM-representable mind.
- Simple and appealing condition vs properties of convergence to knowledge.

Learners taking a more prescribed course of action by basing their conjectures on symptoms (DFTTs).

Preset learning

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Objection: infinite collections of DFTTs.
Solution: f_{dftt}, which for a finite X and s_i says if X is a DFTT of s_i.

If (S, Φ) is conclusively learnable
then there is f_{dftt} that for each world recognizes at least one DFTT.

Preset learning: some results

1. conclusive learnability = preset conclusive learnability
2. preset learners are exactly those that react solely to the content

Fastest learner:

conclusively learns a world s_i as soon as objective 'ambiguity' disappears;

settles on the right world as soon as any DFTT for it has been given.

Definition

(S, Φ) is *conclusively learnable in the fastest way* if and only if there is a learning function L such that, for each ε and for each $i \in \mathbb{N}$,

$$L(\varepsilon{\restriction}n) = i \quad \text{iff} \quad \exists D_i^j \in \mathbb{D}_i \; (D_i^j \subseteq \mathsf{set}(\varepsilon{\restriction}n)) \; \& \\ \neg\exists D_i^k \in \mathbb{D}_i \; (D_i^k \subseteq \mathsf{set}(\varepsilon{\restriction}n-1)).$$

Such L is a *fastest learning function*.

THEOREM

There is a uniformly decidable epistemic space that is conclusively learnable, but is not conclusively learnable in the fastest way.

fastest conclusive learnability is properly included in conclusive learnability

Conclusions

- Complexity of learning/teaching strategies in conclusive learning.
- Complexity of min-DFTT and min-size DFTT related concepts.
- The notion of preset learner in fconclusive learning.
- Not all conclusively learnable classes are learnable in the fastest way.
- We have established a new, more restrictive kind of learning.

even if computable convergence to certainty is possible
it may not be computably reachable just when objective ambiguity disappears

Thank you!

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Proof I

Definition (Smullyan 1958)

Let $A, B \subset \mathbb{N}$. A separating set is $C \subset \mathbb{N}$ such that $A \subset C$ and $B \cap C = \emptyset$. In particular, if A and B are disjoint then A itself is a separating set for the pair, as is B. If a pair of disjoint sets A and B has no computable separating set, then the two sets are *computablely inseparable*.

Let A and B be two disjoint r.e. computablely inseparable sets, such that:

- $x \in A$ iff $\exists y\, Rxy$ with R computable, and
- $x \in B$ iff $\exists y\, Sxy$ with S computable.

For each x there is at most one y, s.t. Rxy and at most one y, s.t. Sxy.
We define $(S_i)_{i \in \mathbb{N}}$:

$$S_i = \{2i, 2i+1\} \cup \{2j \mid Rji\} \cup \{2j+1 \mid Sji\}.$$

Proof II

The idea is that $S_i = \{2i, 2i+1\}$ except that, for some m, Rim or Sim may be true, and then $2i \in S_m$ or $2i+1 \in S_m$, respectively.

Note that:

- There can be at most one such m, and for that m only one of Rim or Sim can be true.
- Since A and B are computably inseparable there is no computable f that makes the choice for each i.
- Except for such intruders the languages are disjoint.

Proof III

The argument:

- $\{2i, 2i+1\}$ is a DFTT for S_i.
- But, $\{2i+1\}$ is a DFTT for S_i if $i \notin B$, and $\{2i\}$ is a DFTT for S_i if $i \notin A$.
- However, a computable function that would give the minimal DFTTs of S_i gives a computable separating set of A and B.
- And this is impossible, since A and B are computably inseparable.

So there cannot be a computable fastest learner!