Deludedly Agreeing to Agree

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A Story

Consider the following story of asymmetric knowledge and trading (a variation on Hart and Tauman (2004)).
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There are two traders, Alice and Bob.

They are no longer talking to each other, but they carefully watch each other's moves in the market.

Private Information

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The private information of Alice and Bob is given by

$$\Pi_A = \boxed{1\quad 2\quad 3\quad 4}\boxed{5\quad 6\quad 7\quad 8}\boxed{9}$$

and

$$\Pi_B = \boxed{1\quad 2\quad 3}\boxed{4\quad 5\quad 6}\boxed{7\quad 8\quad 9}.$$

Signals

In many models, the private information of the traders may come from signals.

Signals

For example, Bob can determine which knowledge partition is the true one based on a set of signals

Signal		States
σ_1	$\rightarrow$	$\{1, 2, 3\}$
σ_2	$\rightarrow$	$\{4, 5, 6\}$
σ_3	$\rightarrow$	$\{7, 8, 9\}$

yielding the partition listed above

$$\Pi_B = \boxed{1 \quad 2 \quad 3}\boxed{4 \quad 5 \quad 6}\boxed{7 \quad 8 \quad 9}.$$

Signal Error

What if Bob makes an error?

Suppose Bob has a defective black box reading signals; when either σ_1 or σ_2 are given as input, the box outputs $4, 5, 6$.

Signal		States
σ_1	$\rightarrow$	$\{4, 5, 6\}$
σ_2	$\rightarrow$	$\{4, 5, 6\}$
σ_3	$\rightarrow$	$\{7, 8, 9\}$

Suppose neither Bob nor Alice knows the black box is defective.

Signal Error

This is bad for Bob – but there is also a dynamic that makes this bad for Alice.

Trading

Event $E = \{4, 9\}$ is a 'good' outcome (e.g., company earnings are about to rise), triggers share purchases.

Suppose true state is 2, and the two traders apply the rule:

$$\begin{cases} \text{Buy} & \text{if the probability of E is 0.3 or more;} \\ \text{Sell} & \text{if the probability of E is less than 0.3.} \end{cases}$$

Trading

On Day 1, Alice, whose partition is

$$\Pi_A = \boxed{1 \quad 2 \quad 3 \quad 4}\boxed{5 \quad 6 \quad 7 \quad 8}\boxed{9}$$

supposes the true state is in $1, 2, 3, 4$.

She judges probability of $E = \{4, 9\}$ is $1/4 \Rightarrow$ seeks to sell shares.

Trading

At true state 2, Bob gets signal σ_1 and erroneously supposes the true state is in $4, 5, 6$,

$$\Pi_B = \boxed{1 \quad 2 \quad 3}\boxed{4 \quad 5 \quad 6}\boxed{7 \quad 8 \quad 9}.$$

He judges the probability of $E = \{4, 9\}$ to be $1/3$, $\Rightarrow$ seeks to buy shares from Alice.

Trading

Since Bob willing to buy on Day 1, Alice ‘learns’ the true state is not in $1, 2, 3$.

Why? Because she does not know that Bob is making mistakes, and she reckons that, recalling the partition

$$\Pi_B = \boxed{1 \quad 2 \quad 3}\boxed{4 \quad 5 \quad 6}\boxed{7 \quad 8 \quad 9},$$

if the true state were in

$$1, 2, 3$$

Bob would judge the probability of $E = \{4, 9\}$ to be zero and would want to sell.

Trading

Remember Alice previously knew the true state is one of

$$1, 2, 3, 4,$$

and now she is convinced that it is not in

$$1, 2, 3.$$

She therefore erroneously supposes on Day 2 that the true state is 4 and offers to buy on Day 2.

Trading

Bob does the same. By Day 3, it is ‘common knowledge’ that 4 is the ‘true state’ – Bob’s error has become Alice’s error!

Now both traders seek to buy as many shares as they can, to their detriment, and a bubble has developed.

Knowledge vs Belief

This motivates our main focus: belief $\neq$ knowledge, and we need to study priors, agreements, etc under *belief*.

- The difference between knowledge and belief comes down to rejecting or accepting what is called the *truth axiom*.

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- Under *knowledge*, the set of states that a player considers possible *always* includes the true state.
- In *belief*, a player may be *deluded*, meaning that he completely disregards the true state, as in the story where true state is 2 but Bob believes that true state is in $\{4, 5, 6\}$. We may think of this as a sort of 'bounded rationality'.

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- In *belief*, a player may be *deluded*, meaning that he completely disregards the true state, as in the story where true state is 2 but Bob believes that true state is in $\{4, 5, 6\}$. We may think of this as a sort of ‘bounded rationality’.
- Axiomatically, knowledge is given in the literature by a system of axioms called **S5** and belief by a system of axioms called **KD45**,

Probabilistic Belief Structure: the Type Function Approach

- A **type function** t_i over Ω for player i assigns for each ω a probability distribution $t_i(\omega) \in \Delta(\Omega)$.
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- This represents player i's beliefs at ω.
- A type function t_i defines a partition Π_i of Ω by $\Pi_i(\omega) = \{\omega' \mid t_i(\omega') = t_i(\omega)\}$; also suppose $t_i(\omega)(\Pi_i(\omega)) = 1$.
- A **probabilistic belief structure** over Ω is a set of type functions $(t_i)_{i \in I}$ over Ω, one for each player in I.

The Possibility Function Approach

- A function $b_i : \Omega \to 2^{\Omega} \setminus \emptyset$ is a *possibility function*.
- The event $b_i(\omega)$ is 'set of states that are considered possible by *i* at ω'; other states are excluded by i at ω.
- If we also have a partition Π_i of Ω, then a possibility function $b_i : \Omega \to 2^{\Omega} \setminus \{\emptyset\}$ such that $b_i(\omega) \subseteq \Pi_i(\omega)$ for each $\omega \in \Omega$ is a *KD45 possibility function*.
- A **belief structure** over Ω is a set of pairs $\Pi = (\Pi_i, b_i)_{i \in I}$, where each b_i is a KD45 possibility function with respect to the partition Π_i of Ω.

$$\boxed{\boxed{1\ 2}\quad 3\quad 4}\boxed{5\quad \boxed{6\quad 7}}\boxed{8\quad \boxed{9}}$$

Inducing belief structures

A probabilistic belief structure $(t_i)_{i \in I}$ over Ω **induces** a belief structure $(\Pi_i, b_i)_{i \in I}$ over Ω, where

- Π_i is the partition of Ω into the types of player i
- $b_i(\omega)$ is the set of states in $\Pi_i(\omega)$ that have positive $t_i(\omega)$ probability.

Example

Consider a probabilistic belief structure over a space of three states

$$t = (0,\ \frac{1}{2},\ \frac{1}{2})$$

This induces the belief structure:

$$b = \boxed{1 \quad \boxed{2 \quad 3}}$$

KD45 Axioms

Given a possibility function b_i, derive a **belief operator** $B_i : 2^\Omega \to 2^\Omega$ by

$$B_i E = \{\omega \mid b_i(\omega) \subseteq E\}. \tag{1}$$

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The following four axioms on a belief operator $B_i : 2^\Omega \to 2^\Omega$ are standard in the literature:

(K) $B_i(\neg E \cup F) \cap B_i E \subseteq B_i F$

(D) $B_i E \subseteq \neg B_i \neg E$

(4) $B_i E \subseteq B_i B_i E$

(5) $\neg B_i E \subseteq B_i \neg B_i E$

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Dov Samet has shown that a belief operator B_i satisfies K,D,4 and 5 if and only if there exists a KD45 possibility function b_i such that the belief operator derivable from b_i is the operator B_i

The Truth Axiom

Belief is 'one axiom short of knowledge': if we add the truth axiom,

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to KD45 then we have the axiom system S5, and we then work with a **knowledge operator** K_i.

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to KD45 then we have the axiom system S5, and we then work with a **knowledge operator** K_i.

The truth axiom states that knowledge is correct; if E is known it must true. This can equivalently be written as $\neg K_i E \cup E = \Omega$.

Delusion

The equivalent concepts in the context of possibility functions are defined in terms of set containment.

Let $\Pi = (\Pi_i, b_i)_{i \in I}$ be a belief structure:

- If $\omega \in b_i(\omega)$ then b_i is **non-deluded** at ω.
- If $\omega \notin b_i(\omega)$ then b_i is **deluded** at ω; i.e. ω is a deluded state for player i.

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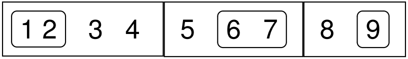

Knowledge Operator

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Then we have

$$K_i(E) = \{\omega \mid \Pi_i(\omega) \subseteq E\}.$$

These are the states at which player i *knows* that E occurs.

For example, suppose that the partition Π_i is

1 2 3 4	5 6 7	8 9

and E is the event $\{6, 7, 8, 9\}$.

Then $K_i(E) = \{8, 9\}$

Belief Operator

Recall that given a KD45 possibility function b_i with respect to a partition Π_i we define a belief operator

$$B_i(E) = \{\omega \mid b_i(\omega) \subseteq E\}.$$

If E is the event $\{6, 7, 8\}$ in this structure

$$\boxed{\boxed{1\ 2}\quad 3\quad 4}\boxed{5\quad \boxed{6\quad 7}}\boxed{8\quad \boxed{9}}$$

then $B_i(E) = \{5, 6, 7\}$.

There is delusion here since $B_i(E) \not\subseteq E$, equivalently, we do not always assume that $\neg B_i E \cup E = \Omega$.

But a player always *believes* that it holds:

Interpersonal Belief Consistency

Intrapersonal Belief Consistency: KD45 $\Rightarrow$ for each event E

$$B_i(\neg B_i E \cup E) = \Omega.$$

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A stronger condition is interpersonal consistency, in which each player believes that not only he, but *all other* players have correct beliefs. **Interpersonal Belief Consistency:**

$$B_i(\neg B_j E \cup E) = \Omega.$$

Non-singular Intuition

Example

Consider:

$$\Pi_A = \boxed{0 \quad \boxed{1 \quad 2 \quad 3 \quad 4}}\boxed{\boxed{5 \quad 6 \quad 7 \quad 8}}\boxed{\boxed{9}}$$

$$\Pi_B = \boxed{0 \quad \boxed{1 \quad 2 \quad 3}}\boxed{\boxed{4 \quad 5 \quad 6}}\boxed{\boxed{7 \quad 8 \quad 9}}.$$

At state 0 the players are **mutually deluded**.

At every other state the players are **mutually non-deluded**.

Singularity

Definition

A state $\omega \in \Omega$ is *non-singular* if it is either mutually deluded or mutually non-deluded. A belief structure is *non-singular* if all the states $\omega \in \Omega$ are non-singular with respect to it.

In words, a belief structure is non-singular if at each state either everybody is right (non-deluded), or everybody is wrong (deluded).

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In words, a belief structure is non-singular if at each state either everybody is right (non-deluded), or everybody is wrong (deluded).

Proposition

A belief structure is non-singular $\Leftrightarrow$ *interpersonal belief consistency is satisfied*

Non-singular Intuition

Looking again at our non-singular example:

$$\Pi_A = \boxed{0 \quad \boxed{1 \;\; 2 \;\; 3 \;\; 4}}\boxed{\boxed{5 \;\; 6 \;\; 7 \;\; 8}}\boxed{\boxed{9}}$$

$$\Pi_B = \boxed{0 \quad \boxed{1 \;\; 2 \;\; 3}}\boxed{\boxed{4 \;\; 5 \;\; 6}}\boxed{\boxed{7 \;\; 8 \;\; 9}}.$$

The state 0 is mutually deluded.

At the state 0, the players both believe that they are living in the following S5 structure:

$$\Pi_A = \boxed{1 \;\; 2 \;\; 3 \;\; 4}\boxed{5 \;\; 6 \;\; 7 \;\; 8}\boxed{\quad 9}$$

$$\Pi_B = \boxed{1 \;\; 2 \;\; 3}\boxed{4 \;\; 5 \;\; 6}\boxed{7 \;\; 8 \;\; 9}.$$

Principle: In non-singular spaces the players 'locally' always believe that they are really living in an S5 structure.

Convictions

For a knowledge operator K_i, the set of all events E that player i knows at ω is called in the literature player i's ken at ω:

$$\mathrm{ken}_i(\omega) \ = \{E \mid \omega \in K_i(E)\}.$$

Convictions

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$$\mathrm{ken}_i(\omega) \ = \{E \mid \omega \in K_i(E)\}.$$

Given a belief operator B_i, call the set of all events E that player i believes at ω player i's **conviction**:

$$\mathrm{con}_i(\omega) \ = \{E \mid \omega \in B_i(E)\}.$$

Con_i will denote the family of all of i's convictions, i.e.,

$$\mathrm{Con}_i \ = \{\mathrm{con}_i(\omega) \mid \omega \in \Omega\}.$$

Analogies from Knowledge to Beliefs

We can study analogues to standard concepts in knowledge spaces in belief spaces by replacing

- Π_i by b_i,
- K_i by B_i
- kens by convictions

Then we can ask: what survives when these replacements are effected? Which properties that hold in S5 are different when considered in the KD45 context?

Credibility

For example, in S5, since it is always true that $\omega \in \Pi_i(\omega)$ for each player i, it follows that $\omega \in \bigcap_{i \in I} \Pi_i(\omega)$.
Hence $\bigcap_{i \in I} \Pi_i(\omega) \neq \emptyset$ for all ω.

In KD45, it may happen that $\bigcap_{i \in I} b_i(\omega) = \emptyset$.

Example

$$\Pi_A = \boxed{1 \quad 2 \quad 3 \quad \boxed{4 \quad 5}}$$

$$\Pi_B = \boxed{\boxed{1 \quad 2} \, 3 \quad 4 \quad 5}.$$

If $\bigcap_{i \in I} b_i(\omega) \neq \emptyset$ for a state ω, we will say that the belief structure is **credible** at that state.

Common Knowledge Component

In S5

- An event F is **common self-evident knowledge** if $\Pi_i(\omega) \subseteq F$ for all $\omega \in F$ and all players i.
- An event E is **common knowledge** at $\omega \in \Omega$ if there is a common self-evident knowledge event F such that $\omega \in F \subseteq E$.
- The smallest event that is common knowledge at a state ω is a **common knowledge component**.

Common Belief Set

Now replace Π_i by b_i. In KD45:

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- An event F is **common self-evident belief** if $b_i(\omega) \subseteq F$ for all $\omega \in F$ and all players i.
- An event E is **common belief** at $\omega \in \Omega$ if there is a common self-evident belief event F such that $\omega \in F \subseteq E$.

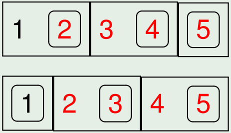
Note that the entire space of states Ω is always common belief by definition.

Denote by $b^Q(\omega)$ the smallest event that is common belief at a state ω – a "common belief set".

Common Belief

Example

Consider, for example



At state 2 there is common belief in the event $F = \{2, 3, 4, 5\}$. We then denote $b^Q(2) = \{2, 3, 4, 5\}$ – the common belief set at 2.

Note that at state 1, Player *A* believes the event *F*, but Player *B* does not. In fact, $b^Q(1) = \{1, 2, 3, 4, 5\}$ – we see that the common belief sets *do not* form a partition.

Common Belief in the Truth

In S5, a common knowledge component at ω can be characterised by

$$\bigcup_{\omega \in T} \Pi_i(\omega) = \bigcup_{\omega \in T} \Pi_j(\omega)$$

for some $T \subseteq \Omega$.

For example

-2 -1	0	□	1 2	3 4	5 6	□	7 8	9
-2	-1 0	□	1	2 3	4 5	6	□	7	8 9

Common Belief in the Truth

Now we replace Π_i's by b_i's, i.e. instead of

$$\bigcup_{\omega \in T} \Pi_i(\omega) = \bigcup_{\omega \in T} \Pi_j(\omega)$$

consider

$$\bigcup_{\omega \in T} b_i(\omega) = \bigcup_{\omega \in T} b_j(\omega). \tag{2}$$

In KD45, the common belief set at a state ω need not necessarily satisfy Equation (2) for some $T \subseteq \Omega$. When this is satisfied, we will want to note it.

Common Belief in the Truth

Definition

There is *strong common belief* in truth at a state ω if there exists $\Omega_0 \subseteq \Omega$ such that $b^Q(\omega) = \bigcup_{\omega' \in \Omega_0} b_i(\omega')$ for all $i \in I$.

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Definition

There is *weak common belief* in truth at a state ω if there exists a state $\omega' \in b^Q(\omega)$ at which there is strong common belief in truth.

Common Belief in the Truth

Proposition

For a belief structure $\Pi = \{\Omega, (b_i)_{i \in I}\}$*, the following are equivalent:*

1. Π *is non-singular (hence it satisfies interpersonal belief consistency).*
2. *For all players* i *and* j, $\bigcap_{\mathbb{C}_i \in Con_i} \mathbb{C}_i = \bigcap_{\mathbb{C}_j \in Con_j} \mathbb{C}_j$.
3. *For all players* i *and* j, $\bigcup_{\omega \in \Omega} b_i(\omega) = \bigcup_{\omega \in \Omega} b_j(\omega)$.
4. *There is strong common belief in truth at every state.*

Delusional Revision

Let μ be a probability distribution over Ω, and let b_i be a belief structure over Ω with corresponding partition Π_i.

We introduce here the **delusional revision** of μ at ω according to b_i, defining it as the probability distribution $\widehat{\mu}(\omega)$ such that

$$\widehat{\mu}(\omega)(\omega') = \begin{cases} \frac{\mu(\omega')}{\mu(b_i(\omega))} & \text{if } \omega' \in b_i(\omega) \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

if $\mu(b_i(\omega)) > 0$; otherwise it is undefined.

Delusional Prior

Let $(t_i)_{i \in I}$ be a probabilistic belief structure over Ω, with $(\Pi_i)_{i \in I}$ the corresponding partition. Let b_i be the belief structure induced by t_i.

A **delusional prior** for t_i is a probability distribution $\mu \in \Delta(\Omega)$, such that $\widehat{\mu}(\omega) = t_i(\omega)$ at each ω, where $\widehat{\mu}(\omega)$ is the delusional revision of μ at ω according to b_i.

Delusional Prior

Example

$$\mu = (1/3, 1/3, 1/3)$$

$$b = \boxed{1 \quad \boxed{2 \quad 3}}$$

The **delusional posterior** is

$$t = (0,\ \frac{1}{2},\ \frac{1}{2})$$

This captures the idea that a player is ‘making a mistake’: he is incorrectly ‘blind’ to state 1, giving it zero probability.

Disagreements

Definition

An n-tuple of random variables $\{f_1, \ldots, f_n\}$ over Ω is a **bet** if $\sum_{i=1}^{n} f_i = 0$.

Let $(t_i)_{i \in I}$ be a probabilistic belief structure and $(b_i)_{i \in I}$ a belief structure induced by $(t_i)_{i \in I}$.

A bet is an **agreeable bet** at ω if $E_i^{t_i}(f \mid b_i(\omega)) > 0$ for all $i \in I$.

It is a **common belief agreeable bet at** ω if it is common belief at ω that f is an agreeable bet.

Common Delusional Prior

The famous Aumann Agreement Theorem states that if there is a common prior and common knowledge of expectations then there can be no common knowledge agreeable bet.

What if we replace common knowledge with common belief and common prior with common delusional prior?
Then the analogous statement does **not** hold.

Common Delusional Prior

Example

$$b_X = \boxed{\boxed{1 \quad 2 \quad 3}}$$

$$b_Y = \boxed{1 \quad \boxed{2 \quad 3}}$$

$$t_X = \frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}$$

$$t_Y = 0\ \frac{1}{2}\ \frac{1}{2}$$

$\mu = (1/3, 1/3, 1/3)$ is a common delusional prior.
Let $H = \{1, 2\}$. The event that player X ascribes probability $\frac{2}{3}$ to H is the entire space.
The event that player Y ascribes probability $\frac{1}{2}$ to H is the entire space.
Therefore, there is common belief of disagreement at every state.

KD45 No Betting

Can we recapitulate something analogous to Aumann's Agreement Theorem in the context of belief?
The answer is **yes**.

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The answer is **yes**.

Theorem

Let $(t_i)_{i \in I}$ be a probabilistic belief structure over Ω and let ω be a state at which there is weak common belief in truth. Then there is a common delusional prior if and only if there is no common belief agreeable bet at ω.

Corollary

In a non-singular belief structure there is a common delusional prior if and only if there is no common belief agreeable bet at any state.

KD45 No Betting

Example

$$t_i = \boxed{\overbrace{1}^{1}}\boxed{\overbrace{2}^{1}}\boxed{\overbrace{3}^{0}\ \overbrace{4}^{0}\ \overbrace{5}^{1}}\boxed{\overbrace{6}^{1/2}\ \overbrace{7}^{1/2}}$$

$$t_j = \boxed{\overbrace{1}^{1/2}\ \overbrace{2}^{1/2}\ \overbrace{3}^{0}\ \overbrace{4}^{0}}\boxed{\overbrace{5}^{1/3}\ \overbrace{6}^{1/3}\ \overbrace{7}^{1/3}}$$

There are an infinite number of common delusional priors; for example,

$$\mu = (\frac{1}{7}, \frac{1}{7}, \frac{1}{14}, \frac{1}{14}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}).$$

There can therefore be no common belief disagreement.

KD45 No Betting

The belief structure induced by the previous example is

$$\boxed{\boxed{1}}\boxed{\boxed{2}}\boxed{3\quad 4\quad \boxed{5}}\boxed{\boxed{6\quad 7}}$$

and

$$\boxed{\boxed{1\quad 2}\quad 3\quad 4}\boxed{\boxed{5\quad 6\quad 7}}$$

At states 3 and 4 the players have no agreement whatsoever in their beliefs (i.e. the belief structure is *not* credible) – $\pi_i(4) = \{5\}$ whilst $\pi_j(4) = \{1, 2\}$ – yet they cannot agree to disagree!

Continuing with the above example

Suppose that we are working in the standard S5 knowledge model and that the players start out with two separate priors, given by

$$\mu_i = (\frac{1}{7}, \frac{1}{7}, \frac{1}{28}, \frac{3}{28}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7})$$

and

$$\mu_j = (\frac{1}{7}, \frac{1}{7}, \frac{1}{14}, \frac{1}{14}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}).$$

Then the players will revise their beliefs into the following posteriors

$$\widehat{t}_i = \boxed{\overbrace{1}^{1}}\boxed{\overbrace{2}^{1}}\boxed{\overbrace{3}^{1/8}\ \overbrace{4}^{3/8}\ \overbrace{5}^{1/2}}\boxed{\overbrace{6}^{1/2}\ \overbrace{7}^{1/2}}$$

$$\widehat{t}_j = \boxed{\overbrace{1}^{1/3}\ \overbrace{2}^{1/3}\ \overbrace{3}^{1/6}\ \overbrace{4}^{1/6}}\boxed{\overbrace{5}^{1/3}\ \overbrace{6}^{1/3}\ \overbrace{7}^{1/3}}.$$

Continuing with the above example

Defining a bet $(f_i, -f_i)$ by

$$f_i = (1/4, 1/4, -6, 3, -1/8, 1/32, 1/32),$$

it can be checked that this bet is common knowledge agreeable at every state under the types $\widehat{t_i}$ and $\widehat{t_j}$.

But if the players make mistakes, wrongly being blind at states 3 and 4, then as we have seen in the example they cannot disagree.

So making mistakes forces them into agreement that they would not have without mistakes!

Thanks

The End

Thanks!